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Mathematical Model of the Simplest Fuzzy PID Controller with Asymmetric Fuzzy Sets

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Abstract: This paper deals with the simplest fuzzy PID controllers which employ two fuzzy sets for each of the three input variables and four fuzzy sets for the output variable. Mathematical model for a fuzzy PID controller is derived by using asymmetric Γ-function type and L-function type membership functions for each input, asymmetric trapezoidal membership functions for output, algebraic product triangular norm, bounded sum triangular conorm, Mamdani minimum inference, nonlinear control rules, and center-of-sums (COS) defuzzification. The effectiveness of the simplest fuzzy PID controller is demonstrated by means of a numerical example along with its simulation results.

1. INTRODUCTION

Proportional-integral-derivative (PID) control is extensively used in industrial control. PI controllers are preferred more to PD controllers as PD controllers are not able to eliminate steady state errors. However, PI controllers show poor performance during the transient state for higher order processes. To obtain overall improved performance, PID controllers are preferred. Conventional PID controllers are usually not effective if the processes to be controlled are higher order and time delay systems, nonlinear systems, complex and vague systems without precise mathematical models, and systems with uncertainties. However, it is apparent from the current literature[5, 8] that fuzzy PI and fuzzy PD controllers can handle the dynamical systems better than their conventional counterparts.

As it appears from the literature, so far two different configurations have been reported for fuzzy PID control as shown in Fig. 1. It was proved [1] that PID controllers could be obtained by using fuzzy control methods like product-sum-gravity method and simplified fuzzy reasoning method . However, PID controller could not be constructed by min-max-gravity method as this method had given a complicated inference result in nonlinear form for a simple fuzzy reasoning, see Fig. 5 in [1].

A fuzzy PID controller structure, based on configuration 1 [2] in Fig. 1, has been proposed. A parameter adaptive method via peak observer has been presented to tune the parameters of the fuzzy controller on-line.

Fuzzy PI and fuzzy PD controllers have been combined to get a fuzzy PID controller according to the configuration 2 [3] in Fig. 1. Its knowledge base consists of two two-dimensional rule bases for PI and PD controls. It has been shown that this fuzzy controller is equivalent to a nonlinear PID controller. A tuning method, based on gain margin and phase margin specifications, has been proposed [4] for determining the parameters of fuzzy PID controller of configuration 2. With this formula, the weighting factors of fuzzy logic controllers can be systematically selected according to the plant under control.

Often fuzzy control applications call for asymmetric input and output fuzzy sets for controlling complex or vague systems. Therefore mathematical models for fuzzy PID controllers with asymmetric input and output fuzzy sets need to be derived.

With the assumption that the membership sum of two neighbouring fuzzy sets is equal to unity, a mathematical model of the simplest fuzzy PID controller (configuration 3 in Fig. 1) is derived by employing algebraic product triangular norm, bounded sum triangular conorm, asymmetric Γ-function type and L-function type membership functions for inputs, asymmetric trapezoidal membership functions for output, nonlinear control rules, Mamdani minimum inference method, and COS method of defuzzification. Simulation results of a numerical example are presented to demonstrate the superiority of fuzzy PID controller over the conventional PID controller.

A broad outline of this paper is as follows: The next section describes the principal components of a typical fuzzy PID controller. Section 3 presents a mathematical model of the simplest fuzzy PID controller with asymmetric fuzzy sets. Section 4 includes simulation results while the last section consists of concluding remarks.

2. COMPONENTS OF A FUZZY PID CONTROLLER

The incremental control signal generated by a discrete-time PID controller is given by

$$\begin{aligned}\Delta u(kT) &= u(kT) - u[(k-1)T] \\ &= K_P^d v(kT) + K_I^d d(kT) + K_D^d a(kT) \qquad (1)\end{aligned}$$

where K_P^d, K_I^d, and K_D^d are respectively the proportional, integral and derivative constants of digital PID controller,

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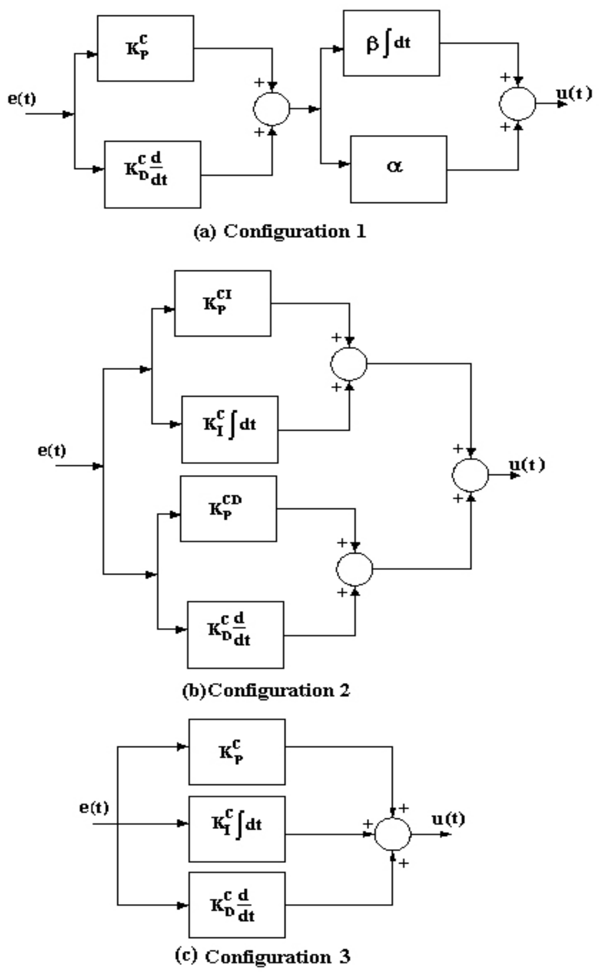

Fig. 1. Fuzzy PID controller configurations

$$v(kT) = \{d(kT) - d[(k-1)T]\}/T, \text{ the velocity} \quad (2)$$

$$d(kT) = e(kT), \text{ the displacement} \quad (3)$$

$$a(kT) = \{v(kT) - v[(k-1)T]\}/T, \text{ the acceleration} \quad (4)$$

$e(kT)$ is the error signal, and T is the sampling period. Eq. (1) is known as 'velocity algorithm' and it is a widely used form of digital PID control. The principal structure of the fuzzy PID controller is shown in Fig. 2 and it consists of the following components.

2.1 Scaling factors

N_d, N_v, N_a and $N_{\Delta u}$ are the normalization factors for the inputs d, v, a, and the output Δu respectively. $N_{\Delta u}^{-1}$ is the reciprocal of $N_{\Delta u}$, called denormalization factor. These scaling factors play a role similar to that of the gain coefficients K_P^d, K_I^d and K_D^d in a conventional PID controller.

2.2 Fuzzification

The fuzzy PID controller employs three inputs: the error signal $e(kT)$ (displacement $d(kT)$), the first-order time derivative of $e(kT)$ (velocity $v(kT)$), and the second-order time derivative of $e(kT)$ (acceleration $a(kT)$).

Let d^*, $v^*$and a^* be the three crisp inputs. Then the fuzzified version of d^* is its degree of membership in $\mu_D(d^*)$, the fuzzified version of v^* is $\mu_V(v^*)$ and the fuzzified version of a^* is $\mu_A(a^*)$ where D, V and A are the linguistic values taken by d_N, v_N and a_N. The inputs are fuzzified by Γ-function type and L-function type membership functions, illustrated in Fig. 3, whose mathematical description is respectively given by

$$\mu_{-X} = \begin{cases} 1, & x_{a_1} \le x \le x_{b_1} \\ -\alpha_x(x - x_{b_2}), & x_{b_1} \le x \le x_{b_2} \\ 0, & x_{b_2} \le x \le x_{a_2} \end{cases} \quad (5)$$

$$\mu_{+X} = \begin{cases} 0, & x_{a_1} \le x \le x_{b_1} \\ \alpha_x(x - x_{b_1}), & x_{b_1} \le x \le x_{b_2} \\ 1, & x_{b_2} \le x \le x_{a_2} \end{cases} \quad (6)$$

$$\text{where} \quad \alpha_x = \frac{1}{x_{b_2} - x_{b_1}} \quad (7)$$

$$\text{Notice that} \quad \mu_{-X} + \mu_{+X} = 1 \quad (8)$$

The fuzzy controller has a single output, called incremental control output $\Delta u(kT)$. The membership functions for the normalized output Δu_N are shown in Fig. 4. The various design parameters x_{a_1}, x_{b_1}, x_{b_2} and x_{a_2} in Fig. 3, and a_-, c_-, d_-, b_-, e, f, a_+, c_+, d_+ and b_+ in Fig. 4 are to be chosen by the designer.

2.3 Control rule base

The following control rules are considered [5] in terms of the abovementioned input and output fuzzy sets.

(R_1) If d_N=$-D$ & v_N=$-V$ & a_N=$-A$ then Δu_N= O_{-2}.
(R_2) If d_N=$+D$ & v_N=$-V$ & a_N=$-A$ then Δu_N=O_{-1}.
(R_3) If d_N=$+D$ & v_N=$-V$ & a_N=$+A$ then Δu_N = O_{+1}.
(R_4) If d_N=$-D$ & v_N=$-V$ & a_N=$+A$ then Δu_N=O_{-1}.
(R_5) If d_N=$-D$ & v_N=$+V$ & a_N=$+A$ then Δu_N=O_{+1}.
(R_6) If d_N=$-D$ & v_N=$+V$ & a_N=$-A$ then Δu_N=O_{-1}.
(R_7) If d_N=$+D$ & v_N=$+V$ & a_N=$-A$ then Δu_N=O_{+1}.
(R_8) If d_N=$+D$ & v_N=$+V$ & a_N=$+A$ then Δu_N=O_{+2}.

where the & symbol in the antecedent part represents

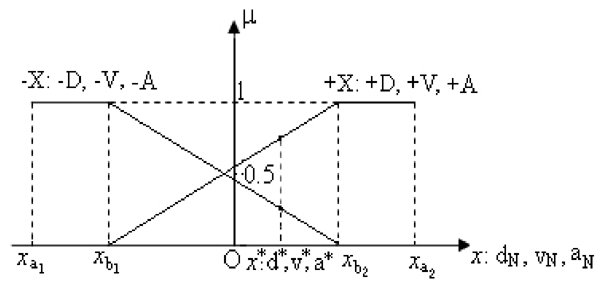

Fig. 3. Fuzzification of crisp values d^*, v^*, and a^*

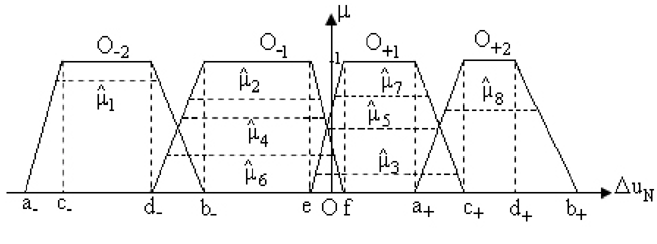

Fig. 4. The output membership functions

the fuzzy "AND" operation which is considered here as

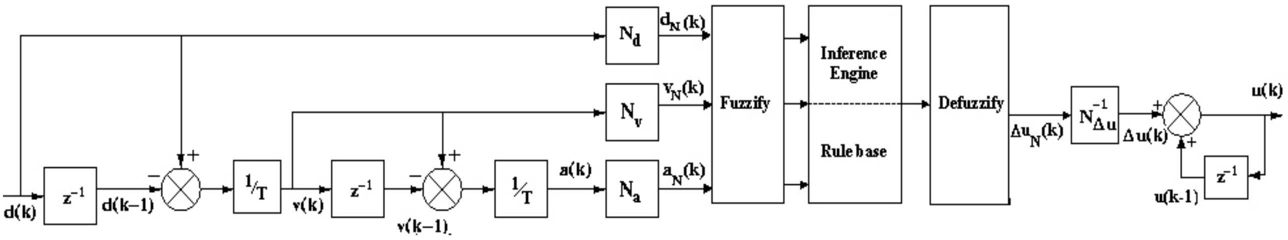

Fig. 2. The fuzzy PID control system

algebraic product triangular norm, and this triangular norm is defined as:

$$\hat{\mu}(d_N, v_N, a_N) = \mu_i(d_N) \cdot \mu_j(v_N) \cdot \mu_k(a_N) \tag{9}$$

where i, j and k are the i^{th}, j^{th} and k^{th} fuzzy sets on d_N, v_N and a_N respectively. Notice that the control rules are nonlinear as the output fuzzy sets are not linearly related to the input fuzzy sets.

2.4 Inference engine

The inference engine first computes the degree of match $\hat{\mu}$ from the crisp input values by using algebraic product triangular norm in Eq.(9). Then the degree of match is used to determine the inferred output fuzzy set using Mamdani minimum inference method, defined as $\min(\hat{\mu}, \mu(\Delta u))$. The reference output fuzzy set (trapezoid), and the inferred output fuzzy set (shown with hatching) are shown in Fig. 5.

As there are three inputs to the fuzzy PID controller,

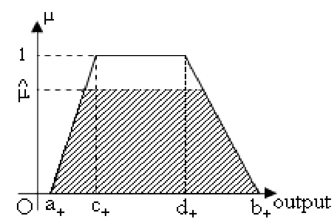

Fig. 5. Mamdani minimum inference method

it is necessary to consider all possible combinations of these variables in a 3D space. A point, say (x_1,y_1,z_1), in a 3D space can always be distinctly shown by taking its projection on the xy-, yz-, and zx- planes. So, as shown in Fig. 6, twenty input combinations are considered in each ($d_N v_N-$, $d_N a_N-$, and $v_N a_N-$) plane so that the state point (d_N^*, v_N^*, a_N^*) can be uniquely located in the 3D cell (subspace) represented by the triplet (n_I, n_{II}, n_{III}) where $n_I, n_{II}, n_{III} = 1, 2, \ldots, 20$. For example, the triplet (9,18,12) represents the 3D cell with 9 from I, 18 from II, and 12 from III of Fig. 6.

The control rules (R_1) to (R_8) of the fuzzy PID controller are used to evaluate appropriate control law in each valid cell (n_I, n_{II}, n_{III}). By using the algebraic product triangular norm the outcome of premise part of each rule is found for all valid cells and is shown in Table 1. There are altogether $20 \times 20 \times 20 = 8000$ cells in the 3D input space. Not all 8000 cells are valid cells; only a few of them are valid. A cell (n_I, n_{II}, n_{III}) is said to be valid if and only if the relations between d_N and v_N, and d_N and a_N produce the relation between v_N and a_N.

It may be seen from the control rules that the output fuzzy sets O_{-1} and O_{+1} are fired three times. In such a situation, bounded sum triangular conorm is used [6, 7] to evaluate combined output fuzzy sets corresponding to the rule sets $\{(R_2), (R_4), (R_6)\}$ and $\{(R_3), (R_5), (R_7)\}$. This triangular conorm is defined as $\min\{\ 1, \mu_A(\Delta u_N) + \mu_B(\Delta u_N)\}$ where A and B are the fuzzy sets on the normalized output Δu_N.

Since the fuzzy controller is having three inputs, when algebraic product triangular norm is used, sum of all the outcomes corresponding to either rule set is less than unity. Therefore the combined membership using bounded sum triangular conorm is given by $\mu(R_2) + \mu(R_4) + \mu(R_6) < 1$ or $\mu(R_3) + \mu(R_5) + \mu(R_7) < 1$

2.5 Defuzzification

The most commonly used COS method is employed to defuzzify the incremental control output. This is expressed as

$$\Delta u_N(kT) = \frac{\begin{array}{c}\{A(\hat{\mu}_1)(h_1) + A(\hat{\mu}_2)(h_2) + A(\hat{\mu}_3)(h_3)\\ +A(\hat{\mu}_4)(h_4) + A(\hat{\mu}_5)(h_5) + A(\hat{\mu}_6)(h_6)\\ +A(\hat{\mu}_7)(h_7) + A(\hat{\mu}_8)(h_8)\}\end{array}}{\sum_{i=1}^{8} A(\hat{\mu}_i)} \tag{10}$$

where $A(\hat{\mu}_i)$ is the area of the inferred output fuzzy set corresponding to the rule R_i and h_i, $i = 1, 2, \cdots, 8$, is the centroid of inferred output fuzzy set (shown with hatching in Fig. 5) corresponding to the rule R_i. As mentioned in Section 2.4, the output fuzzy set O_{-1} is fired three times for the rule set $\{(R_2), (R_4), (R_6)\}$ and O_{+1} is fired three times for the rule set $\{(R_3), (R_5), (R_7)\}$. In this situation, using the bounded sum triangular conorm, Eq.(10) can be written as

$$\Delta u_N(kT) = \frac{\begin{array}{c}\{A(\hat{\mu}_1)(h_1) + A(\hat{\mu}_{2/4/6})(h_{2/4/6})\\ +A(\hat{\mu}_{3/5/7})(h_{3/5/7}) + A(\hat{\mu}_8)(h_8)\}\end{array}}{\begin{array}{c}\{A(\hat{\mu}_1) + A(\hat{\mu}_{2/4/6})\\ +A(\hat{\mu}_{3/5/7}) + A(\hat{\mu}_8)\}\end{array}} \tag{11}$$

where $\hat{\mu}_{2/4/6}$ and $\hat{\mu}_{3/5/7}$ are the outcomes obtained using the triangular conorm. The area of the inferred output

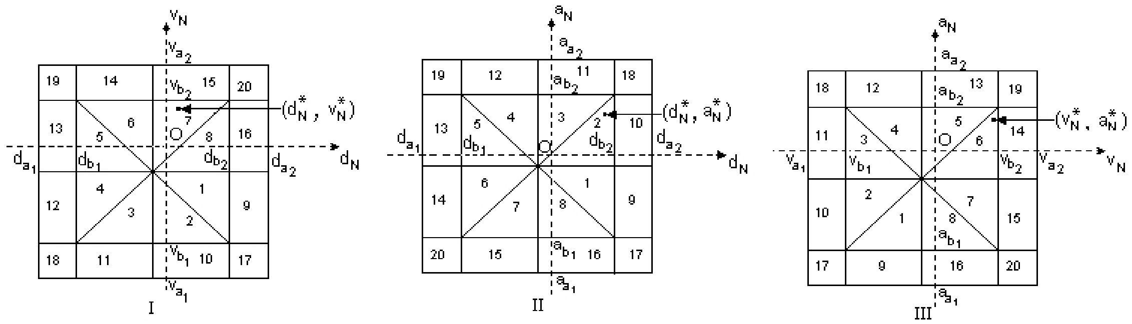

Fig. 6. Regions of fuzzy PID controller input combinations

Table 1. The outcomes of 'algebraic product' operation of premise part of fuzzy control rules $(R_1) - (R_8)$ for valid 3D cells

Cells	(R_1) $\hat{\mu}_1$	(R_2) $\hat{\mu}_2$	(R_3) $\hat{\mu}_3$	(R_4) $\hat{\mu}_4$	(R_5) $\hat{\mu}_5$	(R_6) $\hat{\mu}_6$	(R_7) $\hat{\mu}_7$	(R_8) $\hat{\mu}_8$
(1, 1, 1) to (8, 8, 8)‡	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8
(9, 17, 9), (16, 17, 16)	0	μ_{-V}	0	0	0	0	μ_{+V}	0
(9, 18, 12), (16, 18, 13)	0	0	μ_{-V}	0	0	0	0	μ_{+V}
(10, 11, 18), (11, 12, 18)	0	0	μ_{+D}	μ_{-D}	0	0	0	0
(10, 16, 17), (11, 15, 17)	μ_{-D}	μ_{+D}	0	0	0	0	0	0
(12, 19, 12), (13, 19, 13)	0	0	0	μ_{-V}	μ_{+V}	0	0	0
(12, 20, 9), (13, 20, 16)	μ_{-V}	0	0	0	0	μ_{+V}	0	0
(14, 12, 19), (15, 11, 19)	0	0	0	0	μ_{-D}	0	0	μ_{+D}
(14, 15, 20), (15, 16, 20)	0	0	0	0	0	μ_{-D}	μ_{+D}	0
(17, 9, 10), (17, 10, 11)	0	μ_{-A}	μ_{+A}	0	0	0	0	0
(17, 17, 17)	0	1	0	0	0	0	0	0
(17, 18, 18)	0	0	1	0	0	0	0	0
(18, 13, 11), (18, 14, 10)	μ_{-A}	0	0	μ_{+A}	0	0	0	0
(18, 19, 18)	0	0	0	1	0	0	0	0
(18, 20, 17)	1	0	0	0	0	0	0	0
(19, 13, 14), (19, 14, 15)	0	0	0	0	μ_{+A}	μ_{-A}	0	0
(19, 19, 19)	0	0	0	0	1	0	0	0
(19, 20, 20)	0	0	0	0	0	1	0	0
(20, 9, 15), (20,10,14)	0	0	0	0	0	0	μ_{-A}	μ_{+A}
(20, 17, 20)	0	0	0	0	0	0	1	0
(20, 18, 19)	0	0	0	0	0	0	0	1

where $\mu_1{=}\mu_{-D}\mu_{-V}\mu_{-A}$, $\mu_2{=}\mu_{+D}\mu_{-V}\mu_{-A}$, $\mu_3{=}\mu_{+D}\mu_{-V}\mu_{+A}$, $\mu_4{=}\mu_{-D}\mu_{-V}\mu_{+A}$, $\mu_5{=}\mu_{-D}\mu_{+V}\mu_{+A}$, $\mu_6{=}\mu_{-D}\mu_{+V}\mu_{-A}$, $\mu_7{=}\mu_{+D}\mu_{+V}\mu_{-A}$, and $\mu_8{=}\mu_{+D}\mu_{+V}\mu_{+A}$
‡ only valid cells in (1, 1, 1) through (8, 8, 8)

fuzzy set shown in Fig. 5 is given by $\hat{\mu}(b_+ - a_+)[2 - \hat{\mu}(1 - \theta_{+2})]/2$, and the expression for centroid of the inferred fuzzy set of reference fuzzy set O_{+2} in Fig. 4 is given by

$$h = \frac{\{3(b_+^2 - a_+^2) - 3\hat{\mu}_8[a_+(c_+ - a_+) + b_+ (b_+ - d_+)] - \hat{\mu}_8^2[(c_+ - a_+)^2 - (b_+ - d_+)^2]\}}{3(b_+ - a_+)[2 - \hat{\mu}_8(1 - \theta_{+2})]} \quad (12)$$

$$\text{where} \quad \theta_{+2} = (d_+ - c_+)/(b_+ - a_+) \quad (13)$$

3. MATHEMATICAL MODEL OF THE SIMPLEST FUZZY PID CONTROLLER WITH ASYMMETRIC FUZZY SETS

In the following, mathematical model of fuzzy PID controller is derived by employing asymmetric L-function type and Γ-function type input fuzzy sets and asymmetric trapezoidal output fuzzy sets. Here we consider the sampling time 'kT' to be 'k' for simplicity.

Case (a): $x_{b_1} \le d_N(k),\ v_N(k),\ a_N(k) \le x_{b_2}$

$$\Delta u(k) = \frac{1}{3N_{\Delta u}}\left(\frac{Num}{Den}\right) \quad (14)$$

$$\begin{aligned}\text{where}\ \ Num = {} & 3\{\hat{\mu}_1(b_-^2 - a_-^2) + (\hat{\mu}_2 + \hat{\mu}_4 + \hat{\mu}_6)(f^2 \\ & -d_-^2) + (\hat{\mu}_3 + \hat{\mu}_5 + \hat{\mu}_7)(c_+^2 - e^2) \\ & +\hat{\mu}_8(b_+^2 - a_+^2) - \hat{\mu}_1^2[a_-(c_- - a_-) \\ & +b_-(b_- - d_-)] - (\hat{\mu}_2^2 + \hat{\mu}_4^2 + \hat{\mu}_6^2) \\ & [d_-(b_- - d_-) + f(f - e)] - (\hat{\mu}_3^2 \\ & +\hat{\mu}_5^2 + \hat{\mu}_7^2)[e(f - e) + c_+(c_+ - a_+)] \\ & -\hat{\mu}_8^2[a_+(c_+ - a_+) + b_+(b_+ - d_+)]\} \\ & -\hat{\mu}_1^3(c_- - a_-)^2 + (\hat{\mu}_1^3 - \hat{\mu}_2^3 - \hat{\mu}_4^3 \\ & -\hat{\mu}_6^3)(b_- - d_-)^2 + (\hat{\mu}_2^3 + \hat{\mu}_4^3 + \hat{\mu}_6^3\end{aligned}$$

$$-\hat{\mu}_3^3-\hat{\mu}_5^3-\hat{\mu}_7^3)(f-e)^2+(\hat{\mu}_3^3+\hat{\mu}_5^3+\hat{\mu}_7^3-\hat{\mu}_8^3)(c_+-a_+)^2+\hat{\mu}_8^3(b_+-d_+)^2$$

$$\text{and}\quad Den=\hat{\mu}_1(b_--a_-)[2-\hat{\mu}_1(1-\theta_{-2})]+(f-d_-)[2(\hat{\mu}_2+\hat{\mu}_4+\hat{\mu}_6)-(\hat{\mu}_2^2+\hat{\mu}_4^2+\hat{\mu}_6^2)(1-\theta_{-1})]+(c_+-e)[2(\hat{\mu}_3+\hat{\mu}_5+\hat{\mu}_7)-(\hat{\mu}_3^2+\hat{\mu}_5^2+\hat{\mu}_7^2)(1-\theta_{+1})]+\hat{\mu}_8(b_+-a_+)[2-\hat{\mu}_8(1-\theta_{+2})]$$

with $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$, $\hat{\mu}_4$, $\hat{\mu}_5$, $\hat{\mu}_6$, $\hat{\mu}_7$, $\hat{\mu}_8$ as defined in Table 1,

$$\theta_{-2}=\frac{d_--c_-}{b_--a_-},\ \theta_{-1}=\frac{e-b_-}{f-d_-},\ \theta_{+1}=\frac{a_+-f}{c_+-e} \quad (15)$$

and θ_{+2} is given in Eq.(13)

Case (b): One normalized input is in the interval $[x_{b_1},\ x_{b_2}]$ and the remaining two normalized inputs are not in the interval $[x_{b_1},\ x_{b_2}]$, see Fig. 3.

For cells (10, 11, 18), (11, 12, 18), (14, 15, 20),(15, 16, 20):

$$\text{In Eq. (14)}\quad Num=3\{\mu_{-D}(f^2-d_-^2)+\mu_{+D}(c_+^2-e^2)-\mu_{-D}^2[d_-(b_--d_-)+f(f-e)]-\mu_{+D}^2[e(f-e)+c_+(c_+-a_+)]\}-\mu_{-D}^3[(b_--d_-)^2-(f-e)^2]-\mu_{+D}^3[(f-e)^2-(c_+-a_+)^2]$$

$$\text{and}\quad Den=\mu_{-D}(f-d_-)[2-\mu_{-D}(1-\theta_{-1})]+\mu_{+D}(c_+-e)[2-\mu_{+D}(1-\theta_{+1})]$$

For cells (9, 17, 9),(16, 17, 16), (12, 19, 12), (13, 19, 13):

$$\text{In Eq. (14)}\quad Num=3\{\mu_{-V}(f^2-d_-^2)+\mu_{+V}(c_+^2-e^2)-\mu_{-V}^2[d_-(b_--d_-)+f(f-e)]-\mu_{+V}^2[e(f-e)+c_+(c_+-a_+)]\}-\mu_{-V}^3[(b_--d_-)^2-(f-e)^2]-\mu_{+V}^3[(f-e)^2-(c_+-a_+)^2]$$

$$\text{and}\quad Den=\mu_{-V}(f-d_-)[2-\mu_{-V}(1-\theta_{-1})]+\mu_{+V}(c_+-e)[2-\mu_{+V}(1-\theta_{+1})]$$

For cells (17, 9, 10), (17, 10, 11), (19, 13, 14), (19, 14, 15):

$$\text{In Eq. (14)}\quad Num=3\{\mu_{-A}(f^2-d_-^2)+\mu_{+A}(c_+^2-e^2)-\mu_{-A}^2[d_-(b_--d_-)+f(f-e)]-\mu_{+A}^2[e(f-e)+c_+(c_+-a_+)]\}-\mu_{-A}^3[(b_--d_-)^2-(f-e)^2]-\mu_{+A}^3[(f-e)^2-(c_+-a_+)^2]$$

$$\text{and}\quad Den=\mu_{-A}(f-d_-)[2-\mu_{-A}(1-\theta_{-1})]+\mu_{+A}(c_+-e)[2-\mu_{+A}(1-\theta_{+1})]$$

For cells (14, 12, 19), (15, 11, 19):

$$\text{In Eq. (14)}\quad Num=3\{\mu_{-D}(c_+^2-e^2)+\mu_{+D}(b_+^2-a_+^2)-\mu_{-D}^2[e(f-e)+c_+(c_+-a_+)]-\mu_{+D}^2[a_+(c_+-a_+)+b_+(b_+-d_+)]\}-\mu_{-D}^3[(f-e)^2-(c_+-a_+)^2]-\mu_{+D}^3[(c_+-a_+)^2-(b_+-d_+)^2]$$

$$\text{and}\quad Den=\mu_{-D}(c_+-e)[2-\mu_{-D}(1-\theta_{+1})]+\mu_{+D}(b_+-a_+)[2-\mu_{+D}(1-\theta_{+2})]$$

For cells (9, 18, 12), (16, 18, 13):

$$\text{In Eq. (14)}\quad Num=3\{\mu_{-V}(c_+^2-e^2)+\mu_{+V}(b_+^2-a_+^2)-\mu_{-V}^2[e(f-e)+c_+(c_+-a_+)]-\mu_{+V}^2[a_+(c_+-a_+)+b_+(b_+-d_+)]\}-\mu_{-V}^3[(f-e)^2-(c_+-a_+)^2]-\mu_{+V}^3[(c_+-a_+)^2-(b_+-d_+)^2]$$

$$\text{and}\quad Den=\mu_{-V}(c_+-e)[2-\mu_{-V}(1-\theta_{+1})]+\mu_{+V}(b_+-a_+)[2-\mu_{+V}(1-\theta_{+2})]$$

For cells (20, 9, 15), (20, 10, 14):

$$\text{In Eq. (14)}\quad Num=3\{\mu_{-A}(c_+^2-e^2)+\mu_{+A}(b_+^2-a_+^2)-\mu_{-A}^2[e(f-e)+c_+(c_+-a_+)]-\mu_{+A}^2[a_+(c_+-a_+)+b_+(b_+-d_+)]\}-\mu_{-A}^3[(f-e)^2-(c_+-a_+)^2]-\mu_{+A}^3[(c_+-a_+)^2-(b_+-d_+)^2]$$

$$\text{and}\quad Den=\mu_{-A}(c_+-e)[2-\mu_{-A}(1-\theta_{+1})]+\mu_{+A}(b_+-a_+)[2-\mu_{+A}(1-\theta_{+2})]$$

For cells (10, 16, 17), (11, 15, 17):

$$\text{In Eq. (14)}\quad Num=3\{\mu_{-D}(b_-^2-a_-^2)+\mu_{+D}(f^2-d_-^2)-\mu_{-D}^2[a_-(c_--a_-)+b_-(b_--d_-)]-\mu_{+D}^2[d_-(b_--d_-)+f(f-e)]\}-\mu_{-D}^3[(c_--a_-)^2-(b_--d_-)^2]-\mu_{+D}^3[(b_--d_-)^2-(f-e)^2]$$

$$\text{and}\quad Den=\mu_{-D}(b_--a_-)[2-\mu_{-D}(1-\theta_{-2})]+\mu_{+D}(f-d_-)[2-\mu_{+D}(1-\theta_{-1})]$$

For cells (12, 20, 9), (13, 20, 16):

$$\text{In Eq. (14)}\quad Num=3\{\mu_{-V}(b_-^2-a_-^2)+\mu_{+V}(f^2-d_-^2)-\mu_{-V}^2[a_-(c_--a_-)+b_-(b_--d_-)]-\mu_{+V}^2[d_-(b_--d_-)+f(f-e)]\}-\mu_{-V}^3[(c_--a_-)^2-(b_--d_-)^2]-\mu_{+V}^3[(b_--d_-)^2-(f-e)^2]$$

$$\text{and}\quad Den=\mu_{-V}(b_--a_-)[2-\mu_{-V}(1-\theta_{-2})]+\mu_{+V}(f-d_-)[2-\mu_{+V}(1-\theta_{-1})]$$

For cells (18, 13, 11), (18, 14, 10):

$$\text{In Eq. (14)}\quad Num=3\{\mu_{-A}(b_-^2-a_-^2)+\mu_{+A}(f^2-d_-^2)-\mu_{-A}^2[a_-(c_--a_-)+b_-(b_--d_-)]-\mu_{+A}^2[d_-(b_--d_-)+f(f-e)]\}-\mu_{-A}^3[(c_--a_-)^2-(b_--d_-)^2]-\mu_{+A}^3[(b_--d_-)^2-(f-e)^2]$$

$$\text{and} \quad Den = \mu_{-A}(b_- - a_-)[2 - \mu_{-A}(1 - \theta_{-2})] + \mu_{+A}(f - d_-)[2 - \mu_{+A}(1 - \theta_{-1})]$$

Case (c): Normalized inputs $d_N(k)$, $v_N(k)$, $a_N(k)$ are not in the interval $[x_{b_1}, x_{b_2}]$.
For cells (17, 17, 17), (18, 19, 18), (19, 20, 20):

$$\Delta u(k) = \frac{\{3(f^2 - d_-^2) - 3[d_-(b_- - d_-) + f(f - e)] - [(b_- - d_-)^2 - (f - e)^2]\}}{3(f - d_-)(1 + \theta_{-1})N_{\Delta u}} \quad (16)$$

For cells (17, 18, 18), (19, 19, 19), (20, 17, 20):

$$\Delta u(k) = \frac{\{3(c_+^2 - e^2) - 3[e(f - e) + c_+(c_+ - a_+)] - [(f - e)^2 - (c_+ - a_+)^2]}{3(c_+ - e)(1 + \theta_{+1})N_{\Delta u}} \quad (17)$$

For cell (18, 20, 17):

$$\Delta u(k) = \frac{\{3(b_-^2 - a_-^2) - 3[a_-(c_- - a_-) + b_-(b_- - d_-)] - [(c_- - a_-)^2 - (b_- - d_-)^2]\}}{3(b_- - a_-)(1 + \theta_{-2})N_{\Delta u}} \quad (18)$$

For cell (20, 18, 19):

$$\Delta u(k) = \frac{\{3(b_+^2 - a_+^2) - 3[a_+(c_+ - a_+) + b_+(b_+ - d_+)] - [(c_+ - a_+)^2 - (b_+ - d_+)^2]\}}{3(b_+ - a_+)(1 + \theta_{+2})N_{\Delta u}} \quad (19)$$

4. ILLUSTRATIVE EXAMPLE

Comparison of the performances of linear PID controller and the simplest fuzzy PID controller is done here by considering the following example:
A nonlinear first-order system

$$\dot{y}(t) = y(t) + sin^2(\sqrt{|y(t)|}) + u(t) \quad (20)$$

with step input of magnitude 4 as the reference signal.
Based on the design methodology in Section 4.7 in [9] the controller parameters are selected. For the above process, the values of sampling period T=0.1sec, proportional gain $K_P^d = 1.8T$, integral gain $K_I^d = 1.8T$, derivative gain $K_D^d = 0.008T$, maximum absolute displacement(error) $|d|_{max} = 4$, maximum absolute velocity $|v|_{max} = 4.10676$, and maximum absolute acceleration $|a|_{max} = 994.592$.

For the fuzzy PID controller, the parameters $N_d = 3.0$, $N_v = 3.0$, $N_a = 0.045$, $N_{\Delta u} = 2.1$, and $l = M = 20$ gave rise to the response in Fig. 7, in which peak overshoot $M_p = 0.184\%$, rise time $t_r = 1.164$ sec, and settling time $t_s = 1.6$ sec. Fig. 7 also shows the response with conventional PID controller, in which peak overshoot $M_p = 37.4442\%$, rise time $t_r = 1.0$ sec, and settling time $t_s = 5.6$ sec. Upon comparison, it is evident from the plots that the fuzzy PID controller performs better, demonstrating its superiority over the conventional PID controller.

5. CONCLUSION

In this paper, mathematical model for fuzzy PID controller has been derived analytically using asymmetric L-function

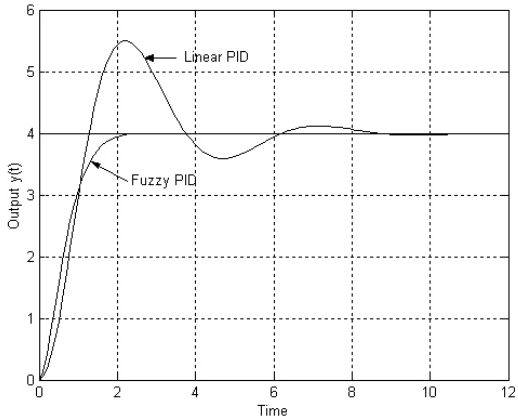

Fig. 7. Step (magnitude 4) response of the closed loop system with nonlinear process

type and Γ-function type input fuzzy sets, asymmetric trapezoidal output fuzzy sets, algebraic product triangular norm, bounded sum triangular conorm, Mamdani minimum inference method and COS defuzzification method. The superiority of fuzzy PID controller over the linear PID controller has been demonstrated through a simulation study on a nonlinear first-order system.

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