Lecture Notes (3)-Principal Stress, Plane and Angle

ENG (NML): 2-854: Mechanical Behaviour of Materials (MBM)

First let us see the definition of Principal Stress, Principal Plane and Principal Angle before we understand and derive the expression for these.

Principal Stress

Principal stresses are maximum and minimum value of normal stresses on a plane (when rotated through an angle) on which there is no shear stress.

Principal Plane

It is that plane on which the principal stresses act and shear stress is zero.

Principal Angle

The orientation of the principal plane with respect to the original axis is the principal angle.

Now our next exercise is to derive an expression for each of these.

In the last lecture, we saw the transformation equations of a stress element as given below

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \qquad ...(1)$$

$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta \qquad ...(2)$$

$$\sigma'_{xy} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \sigma_{xy}\cos 2\theta \qquad ...(3)$$

If we differentiate equation (1) with respect to θ and equate to zero, we have

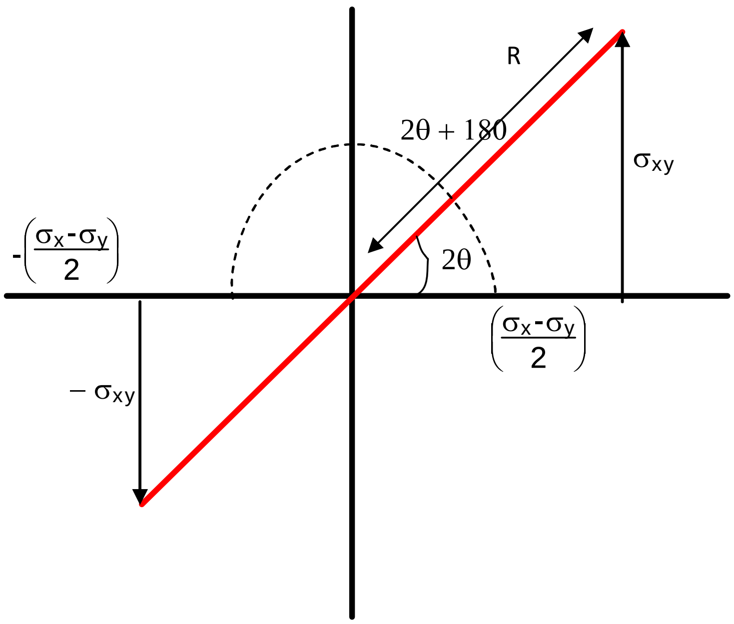
$$\frac{d\sigma'_x}{d\theta} = \frac{\sigma_x - \sigma_y}{2}\,(-\sin 2\theta)\,\cancel{2} + \sigma_{xy}\,(\cos 2\theta)\,\cancel{2} = 0$$

or

$$\tan 2\theta = \frac{\sigma_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)} \qquad \text{or} \qquad \tan 2\theta_p = \frac{\sigma_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)}$$

Note we have put a subscript "p" to the angle to denote that this angle is for principal stress.

This means that if we have a co-ordinate system whose X-axis is $\left(\frac{\sigma_x - \sigma_y}{2}\right)$ and Y



axis is (σ_{xy}) then the plane of consideration will make an angle tan2θ. This is illustrated in the figure below.

We have basically two principal planes, one at angle 2θ and other at $2\theta + 180$. In other words, there are two principal planes one at θ and other at $\theta + 90$.

The next task is to find out the maximum and minimum values of the normal stress on this plane. To do that let us consider the triangle on the right hand side of the above picture and apply Pythagoras Theorem.

Let us assume the hypotenuse of the triangle be R.

Then we can write

$$R^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2 \qquad \text{or} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2}$$

We can also write

$$\cos 2\theta = \left(\frac{\sigma_x - \sigma_y}{2R}\right) \qquad \text{and} \qquad \sin 2\theta = \left(\frac{\sigma_{xy}}{R}\right)$$

Substituting these values of $\cos 2\theta$ and $\sin 2\theta$ in eqn (1)

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} . \frac{\sigma_x - \sigma_y}{2R} + \sigma_{xy} \frac{\sigma_{xy}}{R}$$

$$= \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 . \frac{1}{R} + \frac{\sigma_{xy}^2}{R}$$

Taking 1/R common

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{R}\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2\right]$$

Note that the term within square bracket and underlined by the dotted line is nothing but R^2.

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + R$$

Re-substituting the value of R, we get the max value of normal stress as

$$\boxed{\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2\right]}}$$

The minimum value will be

$$\boxed{\sigma_{min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2\right]}}$$

We have derived the maximum and minimum values of the normal stresses. Since we have assumed that we have rotated the plane through principal angle, these are the maximum and minimum values of the principal stresses. Note that that the principal stresses are denoted as σ_1 (maximum) and σ_2 (minimum)

Now our next task is to find out the maximum value of the shear stress.

Differentiate eqn. (3) and equate it to zero. We get,

$$\frac{d\sigma'_{xy}}{d\theta} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)(\cos 2\theta)\ \cancel{.2} + \sigma_{xy}\ (-\sin 2\theta)\ \cancel{.2} = 0$$

$$= -\left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \sigma_{xy} \sin 2\theta = 0$$

$$-\left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta = \sigma_{xy} \sin 2\theta$$

Since we are dealing with shear, let us denote this angle as θ_s. Then the above equation can be written as,

$$\tan 2\theta_s = -\left(\frac{\sigma_x - \sigma_y}{2\sigma_{xy}}\right)$$

If you observe the equation for θ_p that we derived earlier with this one, the eqn. of θ_s is exactly reciprocal. So we can write,

$$\tan 2\theta_s = -\frac{1}{\tan 2\theta_p} = -\cot 2\theta_p = \tan(90 + 2\theta_p)$$

If we equate the first and last term,

$$2\theta_s = 90 + 2\theta_p$$

or

$$\theta_s = 45 + \theta_p$$

That means the maximum shear stress will always exist at an angle of 45 degree to the principal plane.

We have therefore derived max. & min values of principal stresses, their angles, max. values of shear stress and its orientation with respect to principal planes.