Mathematical Modelling & Its Application To Solidification In A Vertical Continuous Casting Mould

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Abstract

Various features of a mathematical model are described along with its limitations and advantages. Essential steps to develop a mathematical model for a physical problem are stated. To identify independent parameters controlling the process involved in the physical problem, use of similarity and dimensional analyses is presented. Methods of solutions applied particularly to conduction with and without phase-change are suggested.

A mathematical model is developed for solidification with air-gap formation in the vertical mould of a continuous casting machine. Similarity analysis predicts that this model is controlled by two independent parameters, the Stefan number, S_t and the ratio of densities of solid liquid phases, ρ_{sl}. The governing equation along with nonlinear boundary condition due to moving solid-liquid interface is solved using the heat balance integral and the quasi-steady methods. They yielded closed form solutions for solidified-shell thickness, air-gap and mould wall temperature in terms of these two parameters. The validity of the model is confirmed with the experimental results and theoretical analysis for no air gap formation in the continuous casting.

1.0: Introduction

Design, control and optimisation of metallurgical processes in the past were based on empirical models. Their empirical relations were derived from observation of the processes, experimental data and experience of working on the processes. They, however, are restricted to the range of experimental data, fail to provide realistic prediction upon extrapolation, cannot be generalised and may not always help to understand the fundamentals of the processes. Mathematical models, therefore, gained recognisation and become important in recent years. They are developed for many metallurgical processes and the present author and co-workers attempted to develop mathematical models for solidification in continuous casting mould[1-3], hot tops for ingot solidification[4], control of movement of electrodes in electron beam melting process[5], chilling process[6], reduction process in retort furnace[7-8], thermomechanical treatment process[9], melting of scrap in B.O.F process[10] and thermal performance of rolls during hot rolling[1]

Mathematical model is a mathematical device which represents a metallurgical or any other process in terms of equations written in algebric, differential or integral form. The model is based on basic principles governing the process. It predict the controlling parameters for which the model is made. But tl prediction may not be always exact since several assumptions have to be made to prepare the model but it is valuable, accurate, realistic and provides prior knowledge for design,

control and optimisation of the process. Conversion of the model data can be adapted to plant and planning of operation of the process becomes easier

This paper describes the step by step procedure for the development of a mathematical model, format of application of similarity analysis for providing independent non-dimensional controlling parameters and methods of solution. Using these steps, a mathematical model for solidification of molten materia in a vertical mould of a continuous casting machine is formulate The air-gap is taken into account in the model. The similarity analysis with suitable and appropriate similarity dimensionless groups is applied to the model. It gives Stephan number, S_t and the ratio of densities of solid and molten phases, ρ_{sl}, as independent controlling parameters. Despite the model is mathema tically nonlinear, employing the heat balances and quasi-steady method, the close form solutions for mould wall temperature, solidified shell thickness and the air-gap thickness are obtaine in terms of stephan number, S_t and ρ_{sl}. Their effects are shown graphically. The validity of the model is confirmed with experi mental data[12], and the theoretical results[1] obtained for no air gap formation and for negligible sensible heat or high latent heat of fusion.

2.0 Mathematical Model

Mathematical Model converts a physical problem associated with a metallurgical or any other process into a set mathematical equations, conditions that bound the process and conditions of starting and or terminating the process. It predicts directly feature of controlling and relevant parameters of the process and does not require the help of experimental data in the prediction.

Following steps are outlined for development of a mathematical model:

* a physical problem encountered in a process is to be described.
* basic and constitutive laws involved in the process are to be identified.
* suitable and realistic assumptions and approximations are to be applied so that the physics of the problem will not be changed. It will yield the idealised problem.
* depending upon the information desired, lumped, differential or integral control volume is to be chosen.
* mathematical equations governing the idealised problem are to be written and initial and boundary conditions associated with the problem are to be prescribed.
* similarity analysis is applied to provide independent parameters controlling the problem of the process.
* suitable mathematical methods are applied to provide solutions in terms of independent parameters.

* results obtained from solutions are to be interpreted with their physical meaning for application to the process.

2.1 Similarity & Dimensional Analyses

Metallurgical and other processes encountered in many industrial applications are described by process control, operating and geometric variables. Their number, called physical variable becomes large. To study their effect upon different characteristics of a process, one variable at a time is changed keeping others invariant. But the effect of these variables may not be always independent of each other. A combination of several variables that will provide an independent effect, is therefore, sought. A process may have one or more than one such combinations. In case of more than one combinations, the effect of one is independent of others and it has advantages of -

* reducing a number of physical variables of a process, to a minimum number of independent parameters,
* representing adequately the characteristics of a process,
* minimising operating variables to predict the characteristics of a process,
* providing independent controlling parameters for a new process,
* facilitating the design of process equipment,
* employing laboratory level data directly to pilot and industrial plants.

Any physical law governing a process is denoted in a form that becomes independent of a particulars system of units used. As a result, the physical quantities describing the process and

represented in terms of a particular system of units yield several combinations of these quantities that would be independent of these units. They are recognised as nondimensional independent parameters. They are less than the number of physical quantities. Their numbers are determined from the difference of the number of physical variables and the number of primary dimensions representing them.

To estimate the number of nondimensional independent parameters, two methods are available. They are -

* similarity analysis
* dimensional analysis

The concept of similarity analysis uses the criteria of geometric, static, kinematic, dynamic, thermal and chemical similarity whereas dimensional analysis employs the method of indices. The later, however, is not versatile and its use in a mathematical model is often discouraged due to

* prior information needed on the mechanism of a process,
* use of less or more number of physical quantities resulting in dimensionless in-compatibity,
* determination of indices through experimental data,
* empirical correlations restricted within the range of experimental data.

Application of these two analyses to a metallurgical or other processes is shown in Fig. 2·1.

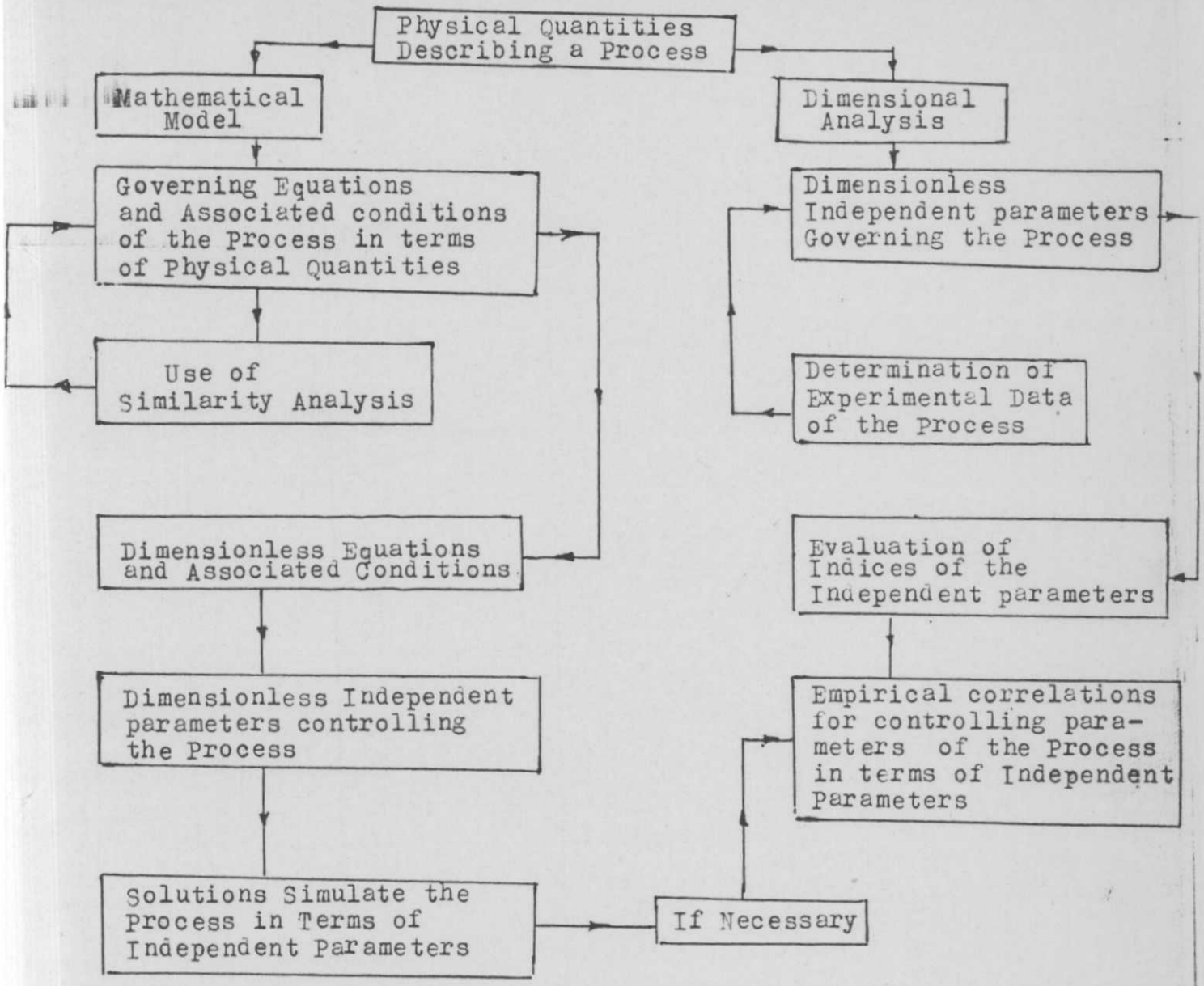

Fig. 2-1

Depending on the type of metallurgical process occured in an industrial plant, the design of its pilot plant needs careful simulation of the metallurgical process. This is possible once different similarity criteria are accounted for by the process. Between a pilot plant and its model, geometric similarity provides a fixed ratio between their corresponding dimensions; kinematic similarity gives a constant value for the ratio of velocities at their corresponding locations; dynamic similarity yields a fixed ratio of their forces acting

on their corresponding locations and times; thermal similarity requires the same temperature gradient in them and chemical similarity states that the rate of chemical reaction must be same at their corresponding locations and times. Their results are shown in Fig. 2·2.

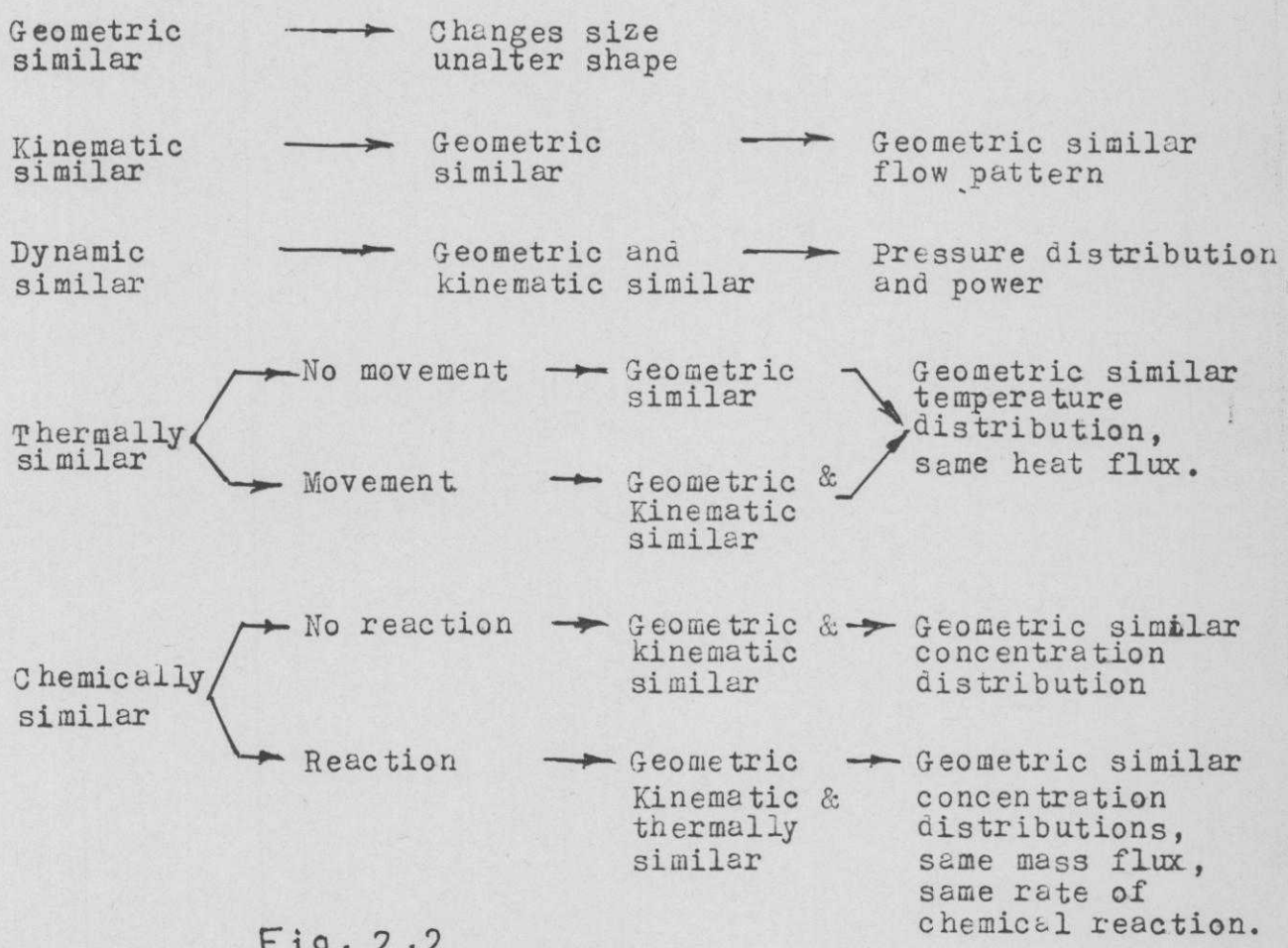

Fig. 2.2

2.2 Method of solutions

A physical problem is transformed into a set of mathematical equations describing the process of the problem and a set of conditions in mathematical form controlling the process. Depending on the type of information needed, the equations can be written in lumped, differential or integral form. Their combinations are also used.

Lumped form mathematical model provides algebric equations that can be readily solved using, matrix inversion, Gaussian elimination, Cramer , iterative and other standard methods. The solution of this form gives the average value of the controlling variable of the process.

In differential formulation, the mathematical model of the process is in terms of differential equations which are complicated and their exact solutions are difficult to obtain. Finite difference and finite element methods are employed. The resulting equations are algebric. Numerical solutions are possible and are applicable in the range for which the solutions are obtained. Semi-analytical methods remove the restriction of application and for many complicated problems lead to closed form solutions applicable to entire range of the process for which the mathematical model is developed. Both numerical and semi-analytical results are accurate and reliable but the accuracy of the former is limited to the selection of mesh or element size and time step whereas the later is accurate near the ~~xxxxxxx~~ controlling boundary. This formulation gives the spacial instant behaviour of the controlling variable of the process.

As the name implied the integral form of mathematical model gives integral equations. These require prescription of the controlling variable that comes within integration. Moreover, the variable must be compatible with the controlling boundaries. This form greatly reduces the complexity of the problem and in many situations it leads to closed-form solutions. Their merits and demerits are just described.

Problems like melting of scrap in B.O.F furnace, ingot solidification, melting of electrodes in electro slag refining process, chill formation during solidification in moulds, continuous casting, ablation, and others encountering melting and solidification are complicated in their mathematical models due to the presence of nonlinearity caused by moving phase-change boundary. Following semi-analytical methods

(i) Biot's variational method

(ii) Heat balance integral method

(iii) Quasi-steady method.

are found to give accurate and reliable results. Moreover, with proper application they often lead to closed form solutions owing to which costly computation and complexity of finite difference and finite element can be avoided.

Biot's Variational Method:[13]

Biot's variational method is based on the principle of irreversible thermodynamics. The heat conduction equation

$$\text{Div } J = -C \frac{\partial \theta}{\partial t} \qquad \ldots \qquad (2.1)$$

$$\text{with } J = -K \text{ Grad } \left(\frac{\partial \theta}{\partial t}\right)$$

is equivalent to a variational equation

$$\int_V C\theta \, \delta\theta \, dv + \delta D = -\int_S J \, \bar{n} . \, \delta\theta \, dS \qquad \ldots \qquad (2.2)$$

where D, is the dissipation function defined as

$$D = \tfrac{1}{2} \int_V K \, (\text{Grad } \theta)^2 \, dv \qquad \ldots \qquad (2.3)$$

In terms of heat flow field, H, the heat conduction equation (2.1) can be written as

$$\text{Div } H = -C \, \theta \qquad \ldots \qquad (2.4)$$

with $H = -K \text{ Grad } \theta$ and $J = \frac{dH}{dt} = \dot{H}$

The variational form of this Eq.(2.4) becomes

$$\int_V C\,\theta\,\delta\theta\,dv + \delta D = -\int_S \theta\,\bar{n}\,.\,\delta H\,dS \quad \ldots. \quad (2.5)$$

where the dissipation function D, is

$$D = \frac{1}{2K}\int_V \left(\frac{dH}{dt}\right)^2 dv \quad \ldots. \quad (2.6)$$

Note that Eq.(2.2) is expressed in terms of temperature field, θ and Eq.(2.5), in terms of heat flow field, H. The selection of Eq.(2.2) or Eq.(2.5) depends on the physics of the problem and the experience of the investigators. For a given physical problem heat transfer and the associated temperature field which are functions of time and space coordinates may not necessarily represent the complete behaviour of the problem. A certain number of parameters q_i, i = 1,2, ...n which are unknown functions of time are taken in the variational method in addition to space coordinates. These parameters, called generalised coordinates, are capable of describing completely the behaviour of the problem. Defining the temperature field as a functions of these

$$\theta = \theta\,(x, y, z, t, q_1, q_2, \ldots, q_n) \quad \ldots \quad (2.7)$$

Eqs.(2.2) and (2.5) can be converted into equations of Lagrangian form. In Eq. (2.7), x,y,z, denote cartesian coordinates.

Using Eq.(2.7), Eq.(2.2) takes the form

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial q_i} = Q_i, \qquad i = 1,2,3, \ldots.n \quad \ldots.. \quad (2.8)$$

with V, the thermal potential

$$V = \frac{1}{2}\int_V C\left(\frac{d\theta}{dt}\right)^2 dv \quad \ldots. \quad (2.8a)$$

and Q_i, the thermal force

$$Q_i = -\int_S J.\,\bar{n}\left(\frac{\partial\theta}{\partial q_i}\right) dS \quad \ldots. \quad (2.8b)$$

$\bar{n}$ is outward normal unit vector.

The heat flow field, H can also be represented in generalised coordinates

$$H = H\,(x, y, z, t, q_1, q_2, \ldots\ldots q_n) \quad \ldots \quad (2.9)$$

Application of this to Eq.(2.5) yields

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial q_i} = Q_i \quad , \quad i = 1,2,3 \ldots\ldots, n \quad (2.10)$$

where the thermal Potential, V is

$$V = \tfrac{1}{2} \int_v C\,\theta^2 \; dv \quad \ldots\ldots \quad (2.10a)$$

and the thermal force

$$Q_i = - \int_S \left(\frac{\partial H}{\partial q_i}\right) . \; \bar{n} \; dS \quad \ldots\ldots \quad (2.10b)$$

In these equations v is volume, S, surface area, k, thermal conductivity and c heat capacity of the solid material.

To employ this method, a suitable temperature profile in terms of generalised coordinate is chosen. It is applied to Eq. (2.1) to estimate J or to Eq.(2.5) to provide, H. Once J and θ are known, Eq.(2.8) can be readily used to give the behaviour of θ and generalised coordinate, q_i . Alternatively, H and θ are used in Eq.(2.10) to predict the behaviour of θ and generalised coordinates.

Heat Balance Integral Method [14]

This method converts the heat conduction equation of partial differential form

$$\nabla^2 \theta = \frac{1}{\alpha} \frac{\partial \theta}{\partial t} \quad \ldots\ldots \quad (2.11)$$

into the integral form of equation. The integration is taken in space or time coordinate. In metallurgical processes where heat conduction occurs with ar without phase-change, Eq.(2.11) is

is idealised as unidirectional

$$\frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n \frac{\partial \theta}{\partial r}\right) = \frac{1}{\alpha} \frac{\partial \theta}{\partial t} \quad \ldots \qquad (2.12)$$

where $n = 0, 1, 3$ denote, respectively, for plate, cylindrical and spherical type of bodies. It is integrated over the special coordinates, r.

$$\int_{r_i(t)}^{r(t)} \frac{\partial}{\partial r}\left(r^n \frac{\partial \theta}{\partial r}\right) dr = \frac{1}{\alpha} \int_{r_i(t)}^{r(t)} \frac{\partial}{\partial t}\left(r^n \theta\right) dr \qquad (2.13)$$

Using Leibnitz theorem, it becomes

$$\left[r^n \frac{\partial \theta}{\partial r}\right]_{r=r(t)} - \left[r^n \frac{\partial \theta}{\partial r}\right]_{r=r_i(t)} = \frac{d}{dt} \int_{r_i(t)}^{r(t)} r^n \theta \, dr + \left[r^n \theta\right]_{r=r_i} \frac{dr_i}{dt} - \left[r^n \theta\right]_{r=r(t)} \frac{dr}{dt} \qquad (2.14)$$

Eq.(2.14) is called the heat balance integral equation. It is applied once a suitable temperature profile is prescribed. It provides solution for θ in terms of desired parameters.

Quasi Steady Method [15]:

Quasi-Steady method assumes that at each instant of time steady state condition prevails. It is used to solve the steady state form

$$\frac{1}{r^n} \frac{d}{dr} \left(r^n \frac{\partial \theta}{\partial r}\right) = 0 \qquad (2.15)$$

of Eq.(2.12). It gives behaviour of temperature. The energy balance is then applied over the portion of the body effected by heat conduction and is equated with heat transfer through the boundary of the body. This leads to

$$- k \frac{\partial \theta}{\partial r}\Big]_{r=r_b} = \rho L \frac{dr}{dt} + \frac{d}{dt} \int_{r_b}^{r} c_p \, \theta \, dm \qquad (2.16)$$

In this equation the left hand side denotes heat transfer through boundary surface, $r = r_b$ whereas on the right hand side the first term represents heat absorbed or realeased due to latent heat of the material of the body caused by phase-change and the second term, the time rate of change of sensible heat of the material of the body, r_b to r is the boundaries of the body between which heat conduction occurs. The solution of Eq. (2.16) gives the behaviour of temperature in desired ~~temperature~~ parameters.

3.0: Physical Problem of Continuously Casting in a Vertical Mould[3]:

Consider a molten material which is continuously casted through a vertical mould of a continuously casting machine. It is shown schematically in Fig. 3.1. The molten material enters from the tundis into the vertical mould at a temperature, T_i and the solidified shell with molten core is continuously withdrawn with a velocity U with which the molten material enters the mould. During casting, coolant flows through the coolant channel of the mould in a direction opposite to the direction of withdrawal of the continuously cast material. The solidification begins once the melt comes in contact with the mould wall. This happens due to heat extraction from the melt by the coolant. It begins to develop at the entrance of the mould and increases progressively in thickness as the melt flows through the mould. Due to change in density occured as a result of phase-change during solidification, the volume of the melt decreases in the solidified layer causing a gap between the mould wall and the solidified skin. It is called air gap. This also starts to develop

at the entrance and grows in thickness progressively along with the progressive growth of the solidified skin.

3.1: Assumptions, Approximations and Conditions:

(i) To avoid over flow of the melt from the top of the mould pouring speed of the melt from the tundis equals to the velocity of the withdrawal, U

(ii) The thickness of the solidified skin at the exit of the mould should be sufficient to withstand the ferrostatic pressure of the melt core in order that the withdrawal takes place without rupture.

(iii) The mould casting interface surface is lubricated for casting speed enhancement, reduction of mould surface wear and improvement of surface quality of casting.

(iv) The molten material is assumed to enter the mould at its melting temperature, $T_i = T_m$. It is not superheated since more heat is to be extracted for the growth of same solidified shell thickness as compared with heat extracted from the molten material having no superheat, consequently, the shell thickness is reduced due to which the increase in ferrostatic pressure is so less that chances of break-out is increased.

(v) Solidification is unidirectional. It is normal to the direction of withdrawal due to heat extraction in that direction.

(vi) Thermophysical properties of the continuously cast material except the density which changes due to change in phase of the material from liquid to solid remain uniform during casting.

(vii) The heat transfer from the mould wall to the coolant is only by convection. Since directions of flow of coolant and withdrawal of continuously casting material forming counter flow heat exchanger keeps about a uniform temperature difference, an average heat transfer coefficient, h is assumed.

(viii) To take into account the couple effect of thermal resistances due to air gap, thickness of the mould wall and the lubrication film, and the average heat transfer coefficient for the convection cooling, overall heat transfer coefficient, U_o is taken. It is given by an expression

$$\frac{1}{U_o} = \frac{1}{h} + \frac{b}{k_m} + \frac{1}{k_l} + \frac{a}{k_a}$$

(ix) The mould is vertical and static.

(x) Presence of intermittant contact regions is neglected.

3.2: <u>Concerned Equations</u>:

Solidification of the molten material in the mould occurs and the solidified shell with molten core is continuously withdrawn. This is affected by heat conduction and the heat conduction equation for a differential control volume within such a continuously withdrawn material is

$$\rho_s c_{ps} (\bar{V} \cdot \nabla T) = k \nabla^2 T \qquad (3.1)$$

The heat conducted through the solidified shell is extracted by coolant owing to heat convection

$$Q_{conv} = Q_{cond} \quad \ldots. \quad (3.2)$$

and the air gap formation is controlled by conservation of mass

$$dm = 0 \quad \ldots. \quad (3.3)$$

3.3:Governing Mathematical Equations:

Representing $q_1(t)$ the distance of the solidification front and a, the air gap thickness measured from the mould wall (Fig.3.1), the Eq.(3.1) for continuous casting of the molten material with uniform velocity U takesthe form

$$\rho_s c_{ps} U \frac{\partial T}{\partial z} = k_s \frac{\partial^2 T}{\partial x^2} \quad q_1(t) \leq x \leq a, \; z > 0 \quad (3.4)$$

Eq.(3.4) accounts for the assumptions stated earlier.

Since during solidification the volume of the melt changes due to change in the density of the melt from ρ_l to ρ_s, the mass of the melt remains constant before and after solidification. Application of Eq.(3.3) to this situation yields

$$\rho_l q_1 = \rho_s (q_1 - a) \quad \ldots. \quad (3.5)$$

3.4:Associated Initial & Boundary Conditions:

Initial Condition:

$$T = T_m \;, \; q_1(t) = 0 \;; \quad a = 0 \text{ at } z = 0 \quad (3.6)$$

Boundary Conditions:

Heat conducted through solidified layers is convected through the coolant. Use of Eq.(3.2) leads this condition to

$$k_s \frac{\partial T}{\partial x} = U_o(q_2 - T_c) \quad \text{with} \quad T = q_2 \text{ at } x=a,\ z > 0 \qquad (3.7)$$

Whereas the heat liberated due to instant solidification is conducted through the solidified layer is

$$k_s \frac{\partial T}{\partial x} = -\rho_l \frac{dq_l}{dt} = -\rho_s \frac{d(q_l - a)}{dt}$$

$$\text{with } T = T_m \text{ at } x = q_l,\ z > 0 \qquad (3.8)$$

Note that Eqs.(3.4)to (3.7) form a mathematical model for the solidification of continuous casting in a vertical mould.

3.5: Similarity Analysis :

This model indicates that the solidification depends upon thermophysical properties of the material to be casted, the mould wall, the gases in the air gap, the lubricant film, the rate of withdrawal, the mould wall thickness, the temperature of the melt and the coolant, heat transfer co-efficient. It is difficult to study their effect upon the behaviour of the solidification because they are interrelated. The independent parameters are, therefore, obtained using similarity analysis.

Geometric similarity gives

$$\xi = x/d$$
$$n = q_l/d \qquad (3.9)$$
$$n_a = a/d$$

Where $d = k_s / U_o$

Here, d is not the semi-gap of the mould often used to provide geometric similarity.

Rather it is the significant thickness of the solidified layer

conducting the amount of heat that would be carried away by convection.

Kinematic similarity leads to

$$\tau = \frac{t}{t_r} \quad \text{with} \quad t = \frac{z}{U} \tag{3.10}$$

t is the time of travel of the solidified layer through a distance z in the mould with the withdrawal velocity, U and t_r, the reference time. It is obtained from Eq.(3.4) using the principle of dimensional homogeneity.

Thermal similarity provides

$$\theta = \frac{(T - T_m)}{(T_c - T_m)}$$

$$\theta_2 = \left(\frac{q_2 - T_m}{T_c - T_m}\right) \tag{3.11}$$

Use of Eqs.(3.9) to (3.11) reduces Eq.(3.4) and its associated initial and boundary conditions, Eqs.(3.6) to (3.8), respectively, to

$$\frac{\partial^2\theta}{\partial\xi^2} = S_t \frac{\partial\theta}{\partial\tau} \qquad n \leq \xi \leq n_a \,,\ \tau > 0 \tag{3.12}$$

$$\theta = 0 \,,\quad n = 0 \,,\quad n_a = 0 \,,\ \tau = 0 \tag{3.13}$$

$$\frac{\partial\theta}{\partial\xi} = (\theta_2 - 1),\ \theta = \theta_2 \,,\ \xi = n_a \,,\ \tau \geq 0 \tag{3.14}$$

$$\frac{\partial\theta}{\partial\xi} = -\frac{1}{\rho_{sl}}\frac{dn}{d\tau} = -\frac{d(n-n_a)}{d\tau},\ \theta=0,\ \xi = n,\ \tau > 0 \tag{3.15}$$

whereas Eq.(3.5) takes the form

$$\rho_{sl}(n - n_a) = n \tag{3.16}$$

The dimensional homogeneity of Eq. (3.12) gives

$$t_r = \frac{\rho_{sl} L\, k_s}{U_o^2 (T_c - T_m)} \tag{3.17}$$

Equations (3.12) to (3.16) state that solidification with air-gap in the vertical mould of a continuous casting machine is controlled by two independent dimensionless parameters, the Stephan number, S_t and the ratio of the densities of the continuous cast material in liquid and solid phases,

3.6: Solution :

The mathematical model of this problem in nonlinear owing to the presence of the moving boundary caused by the solidification. This appears in Eq.(3.15). Since the exact analytical method never provides a closed-form solution, the heat balance integral and the quasi-steady methods, capable of giving closed-form solutions are employed.

3.7: Heat Balance Integral Method :

This method converts the governing differential equation, Eq.(3.12) in the integral form with the limit of integral varying from n_a to n.

$$S_t \int_{n_a}^{n} \frac{\partial \theta}{\partial \tau} d\xi = \int_{n_a}^{n} \frac{\partial^2 \theta}{\partial \xi^2} d\xi \qquad (3.18)$$

It leads to

$$S_t \left[\frac{d}{d\tau} \int_{n_a}^{n} \theta \, d\xi \right] = \left[\frac{\partial \theta}{\partial \xi} \right]_{\xi = \bar{n}} - \left[\frac{\partial \theta}{\partial \xi} \right]_{\xi = \bar{n}_a} - S_t \theta_2 \frac{dn_a}{d\tau} \qquad (3.19)$$

Equation (3.19) is known as the heat balance integral equation for the solidification with air gap formation in the vertical mould of the continuous casting. It readily gives solution once a temperature distribution compatible with the boundary conditions. Eqs.(3.14) and (3.15) is prescribed. A linear

temperature profile that satisfies these conditions is taken.

$$\theta = \theta_2 \left(1 - \frac{\xi - n_a}{n - n_a} \right) \tag{3.20}$$

Such a choice is justified because it was demonstrated in earlier studies that this type of profile for phase change energy storage due to melting of semi-infinite energy storing materials occured with the application of different types of heat injection [16-18], and melting problems[19-23] and the solidification in absence of air-gapformation in a continuous casting mould[1] yielded reliable and accurate results.

Using Eq.(3.20) on the left hand side and Eqs.(3.14) and (3.15) on the right hand side of Eq.(3.19) gives

$$S_t \frac{d}{d\tau}\left[\frac{1}{2}\theta_2(n-n_a)\right] = -\frac{d(n-n_a)}{d\tau} - (\theta_2 - 1) - \theta_2 \frac{dn_a}{d\tau}$$

Its simplified form becomes

$$\frac{1}{2} S_t \theta_2 \frac{d(n+n_a)}{d\tau} + \frac{1}{2} S_t (n-n_a)\frac{d\theta_2}{d\tau} + \frac{d(n-n_a)}{d\tau} = -(\theta_2 - 1) \tag{3.21}$$

Note that this equation contains three unknown θ_2, n and n_a. In order to obtain their unique solutions, two more equations are needed. Conservation of mass, Eq.(3.16) is one equation whereas the second equation is given by the combination of Eqs. (3.14) and (3.20).

$$n - n_a = \theta_2 / (1 - \theta_2) \tag{3.22}$$

Substitution of Eq.(3.16) in Eq.(3.22) gives

$$n = -\xi_{sl}\, \theta_2 / (\theta_2 - 1) \tag{3.23}$$

Using Eq.(3.23) to Eq.(3.16), a relation between n_a and θ_2 is obtained.

$$n_a = n(\rho_{sl} - 1)/\rho_{sl} = -(\rho_{sl} - 1)\,\theta_2/(\theta_2 - 1) \tag{3.24}$$

Combination of Eqs.(3.21),(3.22) and (3.24) leads to

$$\left[\tfrac{1}{2} S_t\, \theta_2 \{A - (\theta_2 - 1)\} + 1\right] \frac{d\theta_2}{d\tau} = -(\theta_2 - 1)^3 \tag{3.25}$$

With $A = \rho_{sl}\left[1 + \left(1 - \frac{1}{\rho_{sl}}\right)\right]$

Here, the nonlinear problem of phase-change due to solidification is reduced to a simple initial value problem with the use of the heat balance integral method. Its closed-form solution becomes

$$\tau = \frac{1}{2} S_t \left[\ln(\theta_2 - 1) - (1 - A)\,\theta_2/(\theta_2 - 1) + \frac{A}{2}\left\{ \frac{A}{(\theta_2 - 1)^2} - 1 \right\}\right]$$

$$+ \frac{1}{2}\left[\frac{1}{(\theta_2 - 1)^2} - 1\right] \tag{3.26}$$

It fulfils the initial condition, Eq.(3.13) and predicts the temperature of the mould wall for solidification with and without air gap in a vertical continuous casting machine. The behaviour of the solidification is given by Eq.(3.23) whereas Eq.(3.24) provides the air gap thisckness once the mould wall temperature θ_2 is substituted.

<u>Solidification with no Air Gap in vertical Continous Casting</u>:

In case of no air gap formation, $n_a = 0$. This gives $\rho_{sl} = 1$, $A = 1$ and Eq.(3.26) reduced to

$$\tau = \frac{1}{2} S_t \ln(\theta_2 - 1) + \frac{1}{2}\left(1 + \frac{1}{2} S_t\right)\left\{\frac{1}{(\theta_2 - 1)^2} - 1\right\} \tag{3.27}$$

Solidification of Materials of Negligible Sensible Heat or High Heat of Fusion in Vertical Continuous Casting :

In this situation, negligible sensible heat gives $C_s \to 0$ or very high latent heat provides $L \to \infty$. Either of these two leads to $S_t \to \theta$ and Eq. (3.26) reduces to

$$\tau = \frac{1}{2}\left[\frac{1}{(\theta_2 - 1)^2} - 1\right] \qquad (3.28)$$

Using Eq. (3.23), Eq.(3.28) in terms of solidification thickness becomes

$$\tau = n/\rho_{sl} + \tfrac{1}{2}(n/\rho_{sl})^2 \qquad (3.29)$$

This equation in terms of air gap thickness, n_a, when Eq.(3.24) takes the form

$$\tau = n_a - (n_a/\rho_{sl}) \qquad (3.30)$$

and in absence of air-gap ($\rho_{sl} = 1$), Eq. (3.29) reduces to

$$\tau = n\,(1 + \tfrac{1}{2}\,n) \qquad (3.31)$$

Eq.(3.31) was also reported previously using variational method for aero-dynamic ablation of melting solids[22], and for solidification in vertical continuous casting with no air-gap[1]. This was also obtained by Goodman[24] and recently by Lunardini[25] employing the heat balance integral problem.

3.8: Quasi-Steady Method :

This method assumes that at each instant of time during solidification there exists a steady state condition. As a result, the steady state form of Eq.(3.12) gives the temperature distribution that appears in Eq.(3.20). Using this, the rate of soli-

dification can readily be obtained by equating the rate of heat release due to solidification to the rate of heat carried from the mould surface at any instant of time.

The conservation of energy denoting Eq.(3.12) is

$$\text{Div } H = -\theta \tag{3.32}$$

Using Eq.(3.20), the heat released H becomes

$$H = \tfrac{1}{2} S_t \, \theta_2 (n-n_a) \left\{ 1 - \frac{\xi - n_a}{n - n_a} \right\}^2 + (n-n_a) \tag{3.33}$$

It satisfies the boundary condition, Eq.(3.15). The time rate of heat release from the surface of the mould ($\xi = n_a$) due to solidification to a thickness n at any instant of time is obtained,

$$Q = \left[\frac{dH}{d\tau} \right]_{\xi = n_a} \tag{3.34}$$

Using Eq.(3.33), Eq.(3.34) becomes

$$Q = \tfrac{1}{2} S_t \left[\theta_2 \frac{dn}{d\tau} + \theta_2 \frac{dn_a}{d\tau} \right] + \tfrac{1}{2} S_t (n-n_a) \frac{d\theta_2}{d\tau} + \frac{d}{d\tau}(n-n_a) \tag{3.35}$$

This heat is carried by the coolant. It is estimated by Eq.(3.14).

$$-Q = \left[\frac{\partial \theta}{\partial \xi} \right]_{\xi = n_a} = (\theta_2 - 1) \tag{3.36}$$

combination of Eqs.(3.35) and (3.36) gives

$$\tfrac{1}{2} S_t \left[\theta_2 \frac{dn}{d\tau} + \theta_2 \frac{dn_a}{d\tau} \right] + \tfrac{1}{2} S_t (n-n_a) \frac{d\theta_2}{d\tau} + \frac{d}{d\tau}(n-n_a) = -(\theta_2 - 1) \tag{3.37}$$

Note that this is the same expression obtained in Eq.(3.21) using the heat balance integral method. The other Eqs.(3.22) to (3.24) are also applied to the quasi-steady method. Use of these Eqs. to Eq.(3.37) yields that same solution that appears in Eq.(3.26).

As a result this method gives identical expressions represented by Eqs. (3.27) to Eqs(3.29).

4.0:Results & Discussions:

Mathematical model for solidification in the vertical mould of a continuous casting machine is developed. The solutions of the nondimensional form of the model are obtained using the heat balance integral and the quasi-steady methods. They have provided closed-form expressions for solidified shell thickness, n, air-gap thickness, n_a, and mould wall temperature, θ_2. Although these two methods are based on different principles, they have given their expressions same. They are controlled by two independent parameters, the Stefan number, S_t and ρ_{sl}. Estimated in Table 1 are the values of S_t and ρ_{sl} for some pure metals and alloys. The results for $\rho_{sl} = 1$ represents the behaviour of n and θ_2 for continuous casting with no air-gap.

Shown in Fig32, are the time variant solidified shell thickness, $n-n_a$, air-gap thickness, n_a and mould wall temperature, θ_2 for different values of S_t with ρ_{sl} taken as a parameter. During initial time they rapidly increase but as the time elapses, their rate of progress of thicknesses and the rise in temperature became slow. This is due to the fact that near the inlet of the mould, the temperature gradient is maximum and the effect of resistance due to air-gap is negligible. As the time increases i.e. the molten material moves in the mould, the air-gap and the solidification shell thicknesses increase. They develop considerable resistance to heat flow. This reduces both temperature gradient and rate of solidification.

Examining the effect of S_t upon $n-n_a$, n_a and θ_2 at any ρ_{sl}, Fig32 shows that for $\rho_{sl} = 1.1$, they increase at any time as S_t decreases. However, in the beginning or near the inlet of the mould, the effect of S_t is not significant, but increasing time exhibits the effect significant and increasing difference between values of each of $n-n_a$, n_a and θ_2.

Fig.3·3 illustrates the time variant $n-n_a$, n_a and θ_2 for different values of ρ_{sl} with the Stefan number, S_t taken as a parameter. During initial time they rapidly increase but as the time elapses their rate of progress of thicknesses and the rise in the temperature of the mould wall become slow. Such a feature can be explained with the explanation given in the above paragraph. Similar feature is observed for all values of $\rho_{sl} > 1$ and for $\rho_{sl} = 1$, no air-gap is formed.

While examining the effect of ρ_{sl}, Fig. 3·3 indicates a starking behaviour for $S_t = 0$. Here, $n-n_a$ and θ_2 are independent of ρ_{sl} but the air-gap formation increases with rising ρ_{sl}. For $S_t > 0$, increase in ρ_{sl} decreases both θ_2 and $n-n_a$ and increases the air-gap thickness, n_a.

From designer's point of view, n, and n_a are plotted in Fig34 for different materials to be continuously cast which are denoted by S_t. They are for different times i.e. at different locations of the mould and for different ρ_{sl}. The plot states that the material with smaller S_t and larger ρ_{sl} n and n_a are more than those having larger S_t and smaller ρ_{sl}. Fig.3·5 also corroborates this fact which is displayed to account the effect of ρ_{sl} as S_t changes.

To establish the validity of the present mathematical model, the present results are compared with the experimental data[12] of unidirectional solidification of tin with air as coolant and the theoretical results of Prasad and Jha[1] for no air-gap formation in Figs.3.6 and3.7. The present results for n is in very close agreement with experimental data appeared in Fig.3.7. But the mould wall temperature in Fig3.6 shows lower values than those obtained from experiment[12] and theoretical analysis[1] for no air-gap formation. It appears that the higher wall temperature in the experiment was due to the average temperature of the coolant taken near the wall and the mould wall temperature was estimated by extraploting the temperature distribution established in the solidified shell. It is believed that the agreement would have improved once the ambient coolant temperature and the temperature at the outer surface of the solidified shell were taken.

5.0: Conclusions

Solidification with air-gap formation in the mould of a continuously casting machine is controlled by two independent parameters, namely the Stefan number, S_t and ρ_{sl} . Application of the heat balance integral and the quasi-steady methods reduces its mathematical model which is nonlinear in simple form leading to closed form solutions. They enable to readily predict the growth of solidified shell thickness and the air-gap formation for any material and at any location in the mould.

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Nomenclature :

a = thickness of air gap, m

b = thickness of the mould wall, m

C_s = heat capacity of the solidified material, Kcal/m^3k

h = heat transfer coefficient, Kcal/h m^2k

K_a = thermal conductivity of the air-gap, Kcal/h m k

K_m = thermal conductivity of the mould wall, Kcal/h m k

K_s = thermal conductivity of the solidified material, Kcal/h m k

L = latent heat of solidification, Kcal/kg.

q_1 = distance of the location of the solid-liquid interface measured from the mould wall, m

q_2 = mould wall temperature, ^oK

S_t = stefan number, $C_s(T_c - T_m)/\rho_s L$

t = solidification time, h

T_c = temperature of the coolant, ^oK

T_s = temperature of the solidified shell, ^oK

T_m = solidification temperature of the material, ^oK

U = withdrawal rate, m/h

U_o = overall heat transfer coefficient, Kcal/h m^2K

x = distance at any location within the solidified shell from the mould wall, m

z = distance at any location from the top of the mould in the direction of withdrawal, m

n = nondimensional distance of the location of the solid-liquid interface, $U_o \varepsilon_1 / k_3$

n_a = nondimensional air-gap thickness, $U_o a/k_s$

ξ = nondimensional distance at any location within the solidified shell, $U_o x/k_s$

ρ_l = density of the liquid phase, kg/m^3

ρ_s = density of the solid phase, kg/m^3

ρ_{sl} = ratio of the densities of solid and liquid phases, ρ_s/ρ_l

θ = nondimensional temperature of the solidified shell, $(T_s - T_m) / (T_c - T_m)$

θ_2 = nondimensional mould wall temperature, $(\varepsilon_2 - T_m)/(T_c - T_m)$

τ = nondimensional time for solidification, $U_o^2 (T_c - T_m)/\rho_s L k_s$

TABLE - 1

Metals	C_p Kcal/Kg°C	T_m, °C	L Kcal/Kg	ρ_s Kg/M³	$\rho_s L$	S_t*
Aluminium	0.241	660.1	96.00	2700	1.132	1.47
Brass	[illegible]					
Copper	0.092	1083.4	48.90	8960	1.120	2.04
Iron	0.180	1539.0	65.00	7368	1.0652	4.26
Lead	0.031	327.4	5.89	11680	1.0938	1.72
Magnesium	0.250	649.0	82.20	1740	1.0943	1.97
Nickel	0.105	1455.0	73.80	8900	1.1258	2.07
Tin	0.0542	231.9	14.50	7300	1.0428	0.792
Zinc	0.0925	419.50	24.40	7144	1.0859	1.51

* Based on $T_c = 0°C$

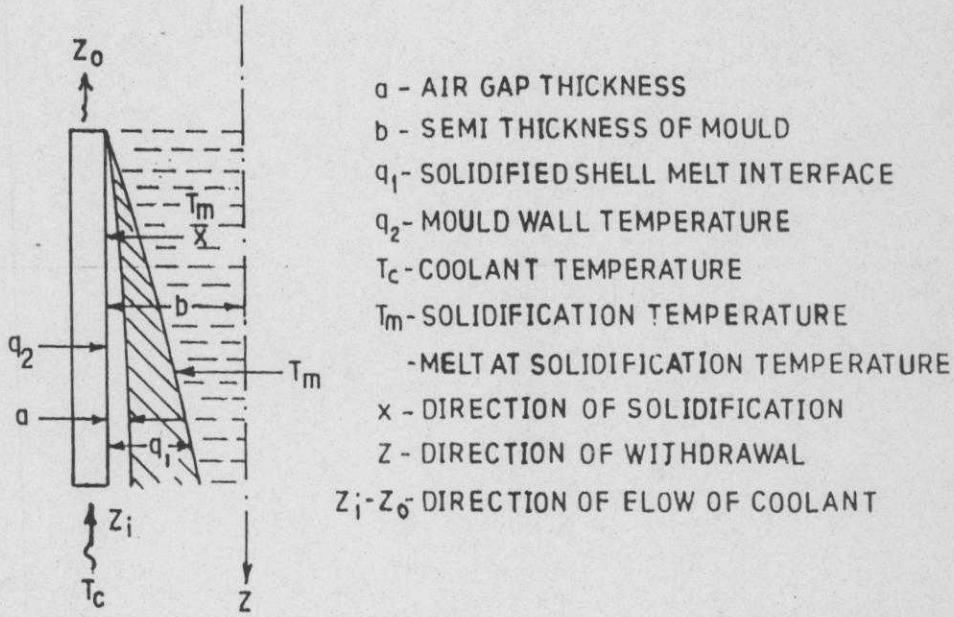

Fig.3.1 Schematric Diagram of Continuous Casting in a Mould.

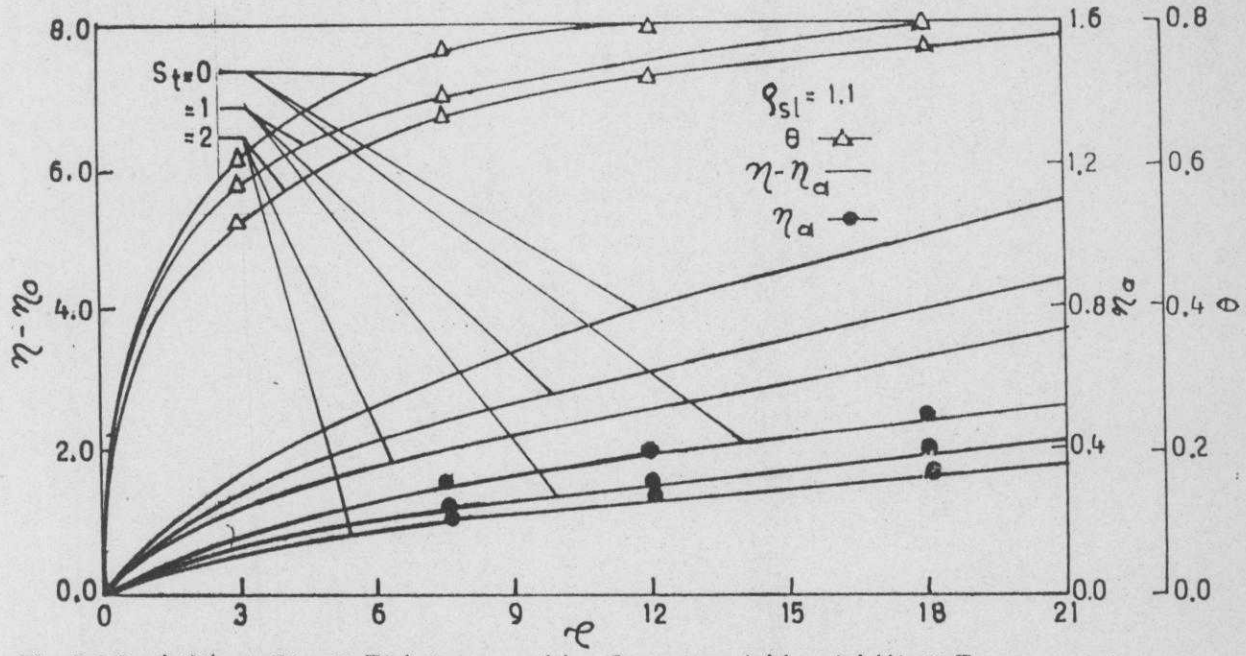

Fig.3.2 Solidified Shell Thickness, Air-Gap and Mould Wall Temperature Time History for Different S_t with a given ρ_{sl}.

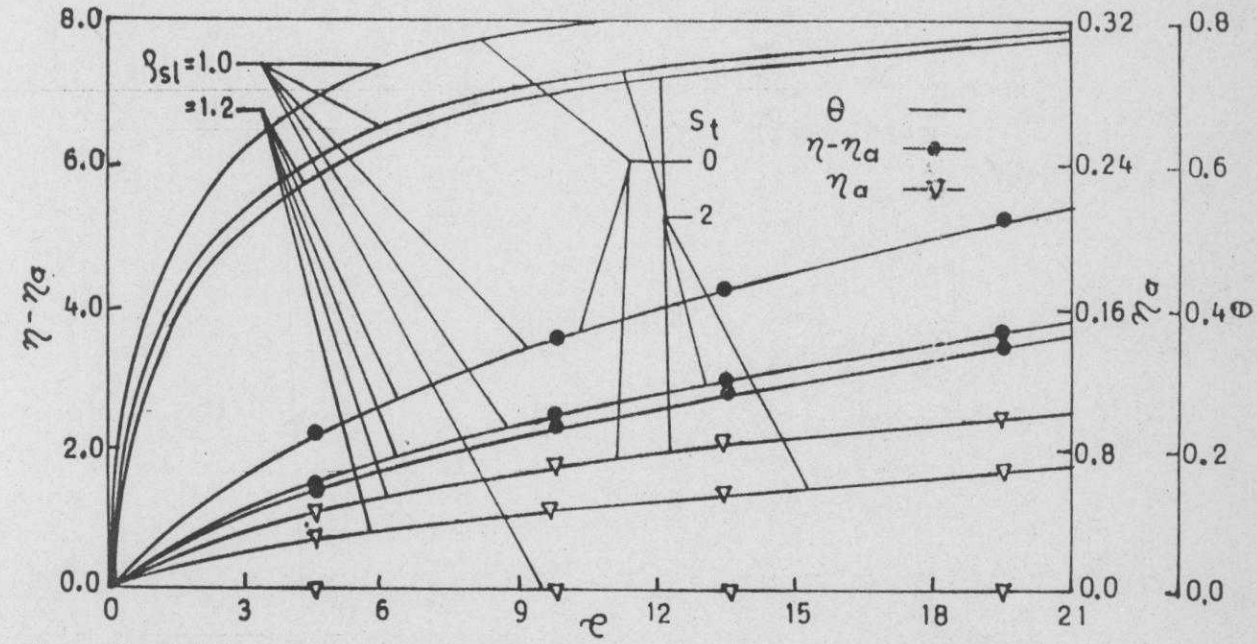

Fig.3.3 Solidified Shell Thickness, Air-Gap and Mould Wall Temperature Time History for Different ρ_{sl} with S_t as a Parameter.

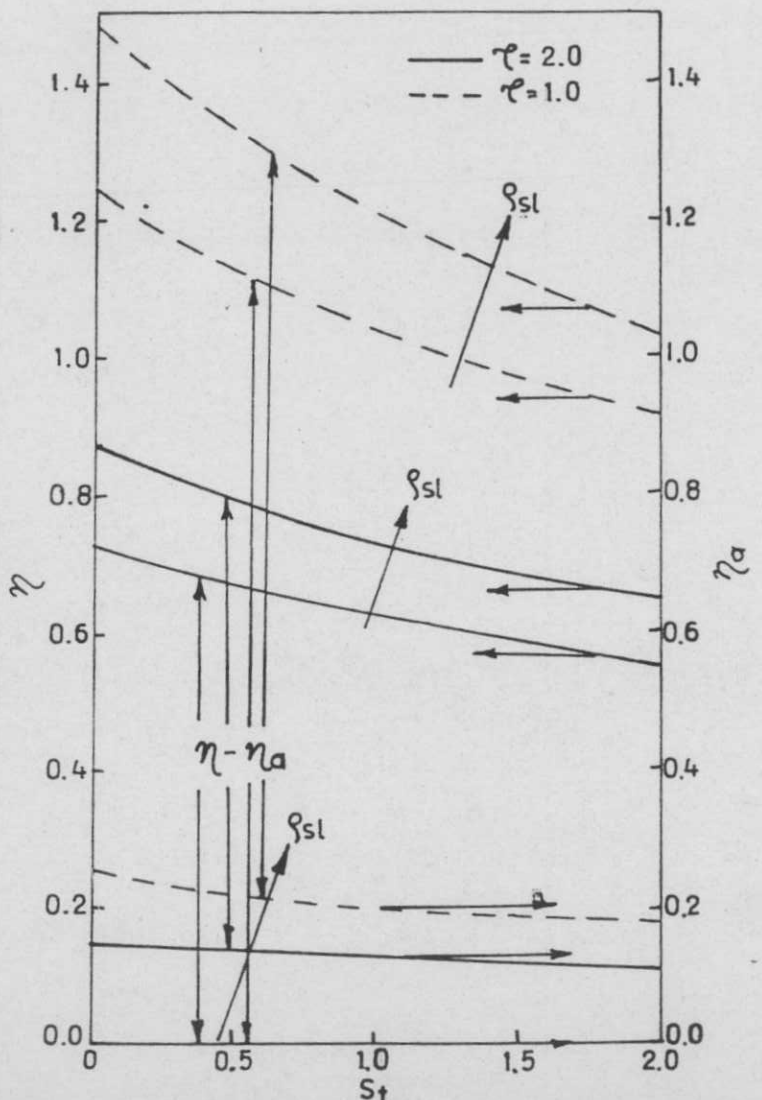

Fig.3.4 Behaviour of η and η_a with S_t for Different ρ_{sl} with τ as a Parameter.

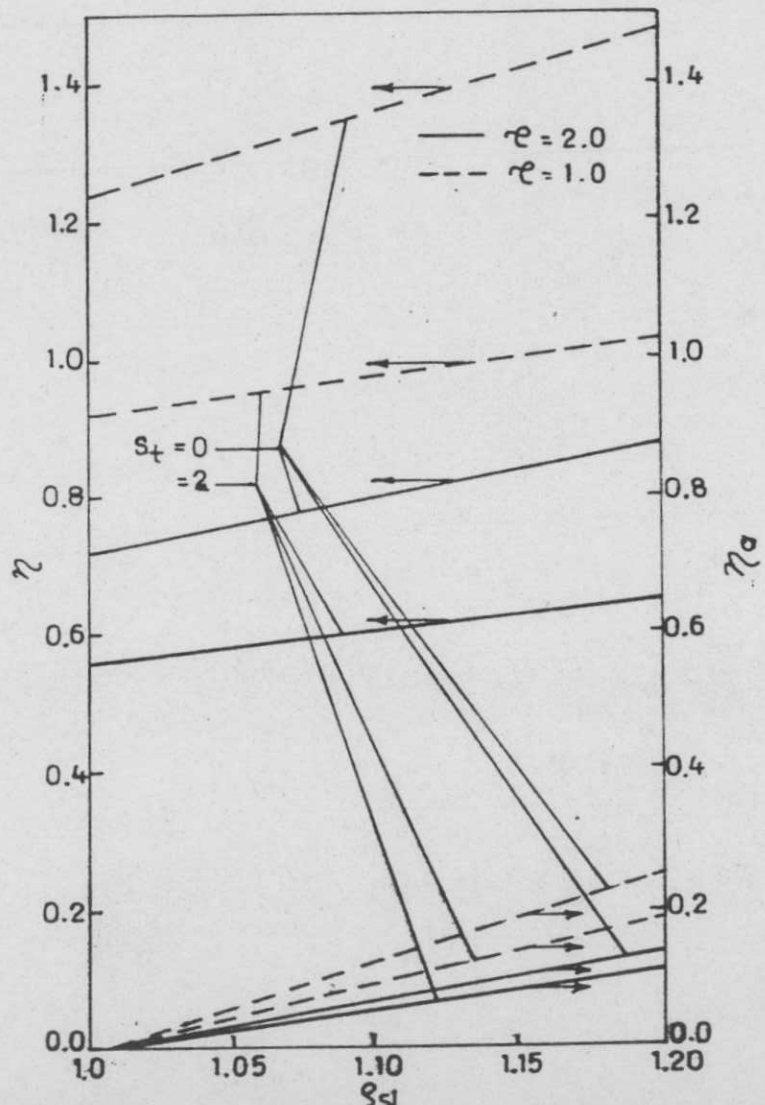

Fig.3.5 Behaviour of η and η_a with ρ_{sl} for Different S_t with τ as a Parameter.

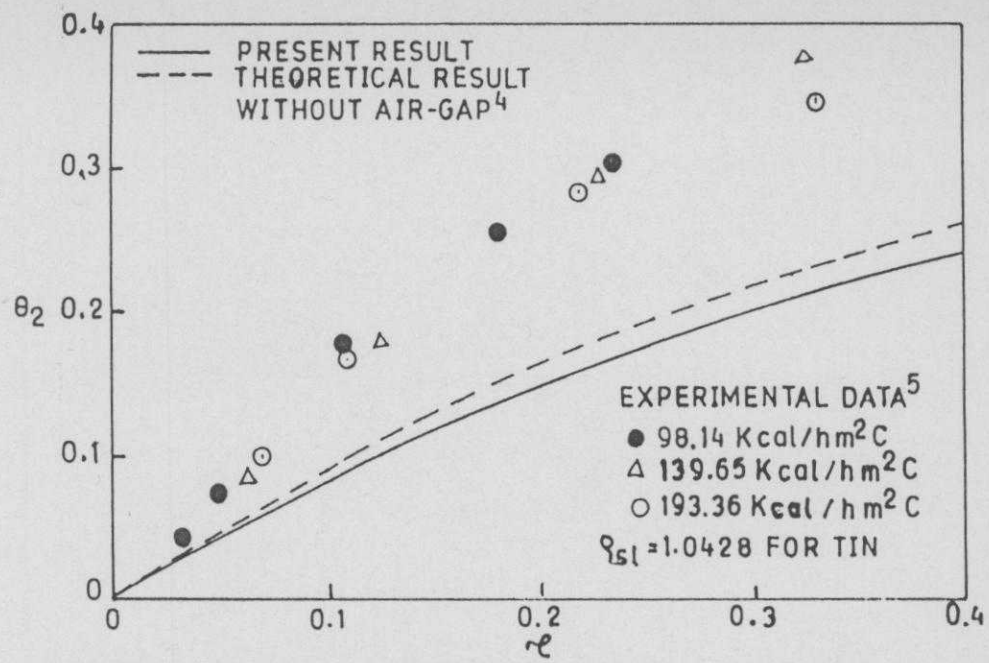

Fig.3.6 Comparison of the Present Results of Mould Wall Temperature with Experimental Data and Theoretical Results of the Past.

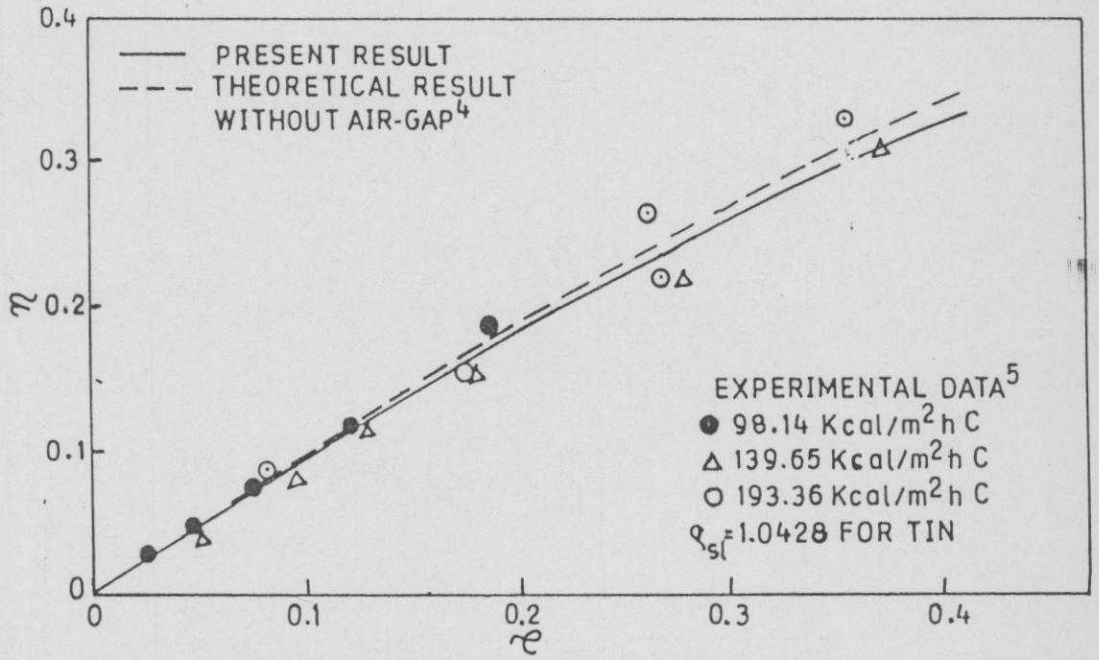

Fig.3.7 Comparison of the Present Result of η with the Experimental and Theoretical Results of the Past.