Analytical Structure and Stability Analysis of a Fuzzy Two-Term Controller with Multi-Fuzzy Sets

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Abstract

This paper reveals the analytical structure of a fuzzy two-term (PI/PD) controller which employs N_1 (≥3) number of symmetric fuzzy sets for the input variable 'displacement', N_2 (≥3) number of symmetric fuzzy sets for the input variable 'velocity' and N_1+N_2-1 number of symmetric fuzzy sets for the output variable 'controller output'. The analytical structures are derived via triangular membership functions for fuzzification of the inputs and output variables, linear control rules, minimum triangular norm, algebraic sum triangular co-norm, different inference methods and center of sums (COS) defuzzification method and properties of such structures are investigated. Using the well-known small-gain theorem, bounded-input bounded-output (BIBO) stability analysis of feedback systems involving fuzzy PD controller as a subsystem is presented. Finally, a numerical example along with its simulation results is included to validate the effectiveness of the fuzzy two-term controller.

1. Introduction

As it appears from the literature, Mizumoto [2] has investigated fuzzy control problem by considering multi-fuzzy sets and different fuzzy reasoning methods. It has been shown [4] that a fuzzy controller can be designed in such a manner that it is at least as good as the conventional PID controller which allows the plant under control to follow a specified behaviour. In [5], a fuzzy PI controller with triangular fuzzy numbers, Zadeh logic to evaluate linear control rules, and center of gravity defuzzifier has been considered. A closed form expression of the defuzzified output has been derived and shown it to be a nonlinear controller. Also, the nonlinearities of the fuzzy controller have been analyzed. Using an arbitrary number of inputs and an arbitrary number of triangular fuzzy sets to fuzzify every input variable, and probability AND operator and Lukasiewicz OR operator to formulate the fuzzy control rules, it has been shown [6] that the defuzzfied output of the controller becomes a linear parametric function of the controller inputs.

A multi-region fuzzy logic controller for nonlinear processes was proposed [7]. Based on apriori knowledge, the process to be controlled was divided into fuzzy regions such as high-gain, low-gain, large-time-constant, and small-time-constant. Then a fuzzy controller was designed based on the regional information. A set of linguistic rules was formulated [8] in terms of their governing equations and asymptotic control laws were proposed for a small input error. Also, the effect of design factors like inference operators and number of partitions were studied. For a single input and single output (linear or nonlinear) system, it was shown [9] that one could construct a fuzzy logic controller equivalent to a given PI controller, and that a fuzzy logic controller designed with prescribed fuzzy logic operations was essentially a PI controller.

The input-output parametric relationship of a class of crisp-type fuzzy logic controllers using various t-norm sum-gravity inference methods has been studied [11]. Using four most important t-norms, the matching level of each control rule has been calculated, and the explicit mathematical forms of reasoning surfaces have been obtained. The reasoning surfaces of these crisp-type fuzzy controllers have been proved to be composed of a 2-D multi-level relay and a local position-dependent nonlinear compensator with output pattern being influenced by the t-norm selected.

Input-output structures of fuzzy controllers with nonlinear input fuzzy sets, singleton output fuzzy sets, product AND operator, Mamdani type of fuzzy rules, and centroid defuzzifier were studied [13]. In addition, conditions for these structures to be equivalent to nonlinear PI/PD controllers with variable gains were provided. Recently, attempts have been made [14] to develop analytical structures for fuzzy PI/PD controllers with arbitrary trapezoidal input fuzzy sets having the property: the sum of two neighboring membership functions is not equal to unity. The analytical structures in [5, 6, 11, 13, 14] have been derived using either maximum or bounded sum triangular conorm.

It is evident from the literature survey that application of algebraic sum triangular co-norm for fuzzy two-term control is yet to be explored. Therefore, the main objectives of this paper are (i) to derive analytical structures of fuzzy two-term (PI/PD) controllers by employing minimum triangular norm, algebraic sum triangular co-norm, N_1 number of symmetric triangular membership functions on the input variable 'displacement', N_2 number of symmetric triangular membership functions on the input variable 'velocity', N_1+N_2-1 number of symmetric triangular membership functions on output, linear control rules, Mamdani minimum / Larsen product / drastic product inference method, and COS method of defuzzification, (ii) to investigate the properties of the controllers derived in (i), and (iii) to establish BIBO stability conditions for the feedback system containing the fuzzy PD controller as the subsystem. This paper is organized as follows: Section 2 deals with scaling factors, fuzzification and defuzzification modules, control rule base, and inference engine; Section 3 presents analytical structures of fuzzy PI controllers with symmetric triangular fuzzy sets; properties of fuzzy PI controllers with symmetric triangular fuzzy sets are discussed in Section 4; Section 5 shows that the analytical structures and their properties presented for fuzzy PI controllers also hold good for fuzzy PD controllers. BIBO stability analysis of fuzzy PD control systems is given in Section 6; Section 7 includes the results of simulation studies done on a linear second-order time delay system. The last section concludes the paper.

2. Fuzzy PI /PD controller

The principal structure of a fuzzy PI / PD controller is shown in Figure 1 which consists of the components such as scaling factors, fuzzification and defuzzification modules, rule base and inference engine. The incremental control signal (velocity algorithm) [3] generated by discrete-time PI controller is given by

$$\Delta u(kT) = u(kT) - u[(k-1)T \qquad = K_P^d\, v(kT) + K_I^d\, d(kT \tag{1}$$

while the control effort produced by discrete-time PD controller is given by

$$u(kT) = K_P^d\, d(kT) + K_D^d v(kT) \tag{2}$$

where K_P^d, K_I^d, and K_D^d are respectively the proportional, integral and derivative constants of discrete-time PI and PD controllers,

$$d(kT) = e(kT), \text{ the displacement} \tag{3}$$

$$\text{and } v(kT) = \frac{\{d(kT) - d[(k-1)T]\}}{T}, \text{ the velocity} \tag{4}$$

$e(kT)$ is the error signal, T is the sampling period. Based on (1), (2) and (4), the principal structures of fuzzy PI controller and fuzzy PD controller are shown in Figure 1, in which N_d, N_v, $N_{\Delta u}^{-1}$ and N_u^{-1} represent scaling factors of the fuzzy controllers, and $d_N(kT), v_N(kT)$, $\Delta u_N(kT)$ and $u_N(kT)$ represent normalized versions of $d(kT)$, $v(kT)$, $\Delta u(kT)$ and $u(kT)$, respectively.

2.1. Scaling factors

The use of normalized universes of discourse requires a scale transformation which maps the physical values of the process state variables (in the present situation $d(kT)$ and $v(kT)$) into a normalized domain. This is called input normalization. Furthermore, output denormalization maps the normalized value of the output variable (here $\Delta u_N(kT)$ or $u_N(kT)$) into its physical domain. The scaling factors which describe the inputs normalization (N_d and N_v) and output denormalization ($N_{\Delta u}^{-1}$ or N_u^{-1}) play a role similar to that of the gain co-efficients (K_P^d, K_I^d, and K_D^d) in conventional controllers.

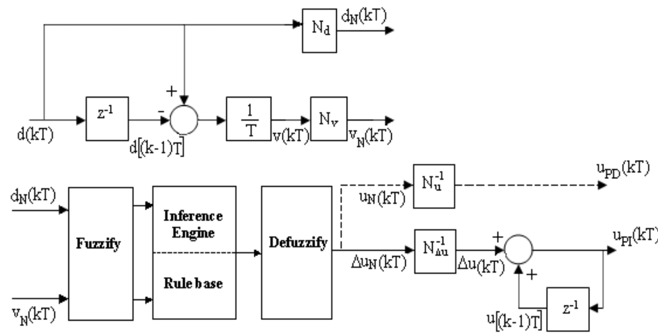

Figure 1. Block diagram fuzzy PI/PD control system

2.2. Fuzzification module

It converts instantaneous value of a process state variable into a linguistic value with the help of represented fuzzy set. The parametric functional description of the triangular shaped membership function is the most economic one and hence it is considered here. Let the number of fuzzy sets, N on scaled input variables "displacement $d_N(kT)$" and "velocity $v_N(kT)$" be the same, and the membership functions be identical. Assume that there are J number of fuzzy sets on negative displacement (velocity), one fuzzy set for zero displacement (velocity), and J number of fuzzy sets on positive displacement (velocity). Therefore, there is a total of

$$N = 2J + 1 \geq 3 \tag{5}$$

number of fuzzy sets on each input variable as shown below:

$$\{X_{-J}, X_{-(J-1)}, \cdots, X_{-1}, X_0, X_1, \cdots, X_p, \cdots, X_{(J-1)}, X_J\} \tag{6}$$

where X is D (for displacement) or V(for velocity). The membership functions corresponding to members in (6) are considered as

$$\{\mu_{-J}(x_N), \mu_{-(J-1)}(x_N), \cdots, \mu_{-1}(x_N), \mu_0(x_N), \mu_1(x_N), \cdots, \mu_p(x_N), \cdots, \mu_{(J-1)}(x_N), \mu_J(x_N)\} \tag{7}$$

where x_N is d_N (with $p = i$) or v_N (with $p = j$). Let the central value of membership function $\mu_p(x_N)$ be λ_p, and define $\lambda_{-J} = -l$, $\lambda_0 = 0$ and $\lambda_J = l$. Also, let the space S between central values of two adjacent members be equal. Then S is given by

$$S = \frac{l}{J} \tag{8}$$

Consequently, the central value λ_p becomes $\lambda_p = p\ .\ S$. Note that the base of each member is $2S$. The membership function $\mu_p(x_N)$ is defined as follows:

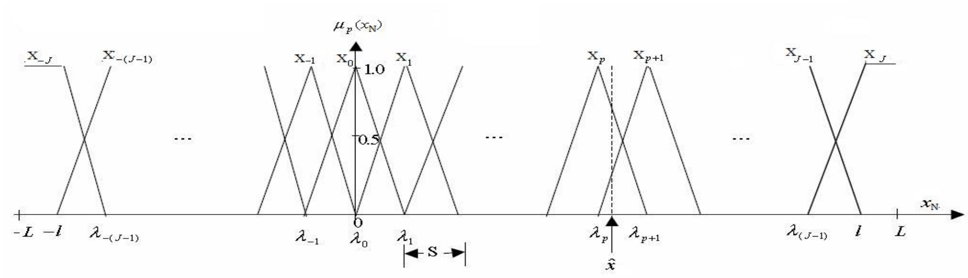

Figure 2. Membership functions for input x_N

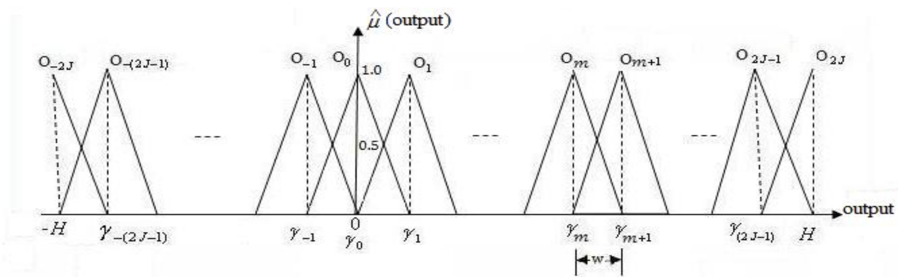

Figure 3. Output membership functions

For $p = -(J-1), -(J-2), \cdots, (J-2)$, and $(J-1)$

$$\mu_i(x_N) = \begin{cases} 0, & x_N \le (p-1)S \\ \frac{1}{S}\{x_N - (p-1)S\}, & (p-1)S \le x_N \le pS \\ \frac{-1}{S}\{x_N - (p+1)S\}, & pS \le x_N \le (p+1)S \\ 0, & (p+1)S \le x_N \end{cases} \quad (9)$$

For $p = -J$

$$\mu_{-J}(x_N) = \begin{cases} 0, & x_N \le -L \\ 1, & -L \le x_N \le -JS \\ \frac{-1}{S}\{x_N - (-J+1)S\}, & -JS \le x_N \le (-J+1)S \\ 0, & (-J+1)S \le x_N \end{cases} \quad (10)$$

For $p = J$

$$\mu_J(x_N) = \begin{cases} 0, & x_N \le (J-1)S \\ \frac{1}{S}\{x_N - (J-1)S\}, & (J-1)S \le x_N \le JS \\ 1, & JS \le x_N \le L \\ 0, & L \le x_N \end{cases} \quad (11)$$

Notice that

$$\mu_p(x_N) + \mu_{p+1}(x_N) = 1, \quad x_N \in [-L, L] \quad (12)$$

$$i + j = m \quad (13)$$

Figure 2 shows membership function $\mu_p(x_N)$ corresponding to the input fuzzy set X_p in (6). Let x_N be the crisp input. Then from Figure 2 the fuzzified version of x_N is its degree of membership $\mu_p(x_N)$. Assume that there are $2N-1$ (i.e. 4J+1) number of fuzzy sets on the normalized output variable $\Delta u_N(kT)$ for PI and $u_N(kT)$ for PD). Among these, 2J members are on negative output, one member for zero output, and 2J members are on positive output. The membership functions for normalized output, shown in Figure 3, are denoted by

$$\{O_{-J}, O_{-(J-1)}, \cdots, O_{-1}, O_0, O_1, \cdots, O_m, \cdots, O_{(J-1)}, O_J\} \quad (14)$$

Let the central value of member O_m be γ_m and define γ_{-2J} =-H, $\gamma_0 = 0$, and γ_{2J} = H. Further, let the space W between the central values of two adjacent members be equal. Then

$$W = \frac{H}{2J} = \frac{H}{N-1} \quad (15)$$

and $$\gamma_m = m.W = \frac{mH}{N-1} \quad (16)$$

2.3. Control rule base

The following linear control rules are considered for fuzzy PI controllers:

R1) If d_N is D_{i+1} AND v_N is V_j then Δu_N is O_{m+1}.
R2) If d_N is D_i AND v_N is V_j then Δu_N is O_m.
R3) If d_N is D_i AND v_N is V_{j+1} then Δu_N is O_{m+1}.
R4) If d_N is D_{i+1} AND v_N is V_{j+1} then Δu_N is O_{m+2}.

The above control rules also hold good for fuzzy PD controller if Δu_N is replaced by u_N. The AND operation considered in the rule base is minimum triangular norm, which is defined as:

$$\hat{\mu}_{min}(d_N, v_N) = \min(\mu_a(d_N), \mu_b(v_N)) \quad (17)$$

where $a \in \{D_i, D_{i+1}\}$ and $b \in \{V_j, V_{j+1}\}$ are the a^{th} and b^{th} fuzzy sets on d_N and v_N respectively.

Table 1. Inference methods

Inference Method	**Definition**	**Area of inferred output fuzzy set A($\hat{\mu}$)**
Mamdani Minimum, R_{MM}	$\min(\hat{\mu}, \mu(output))$	$\hat{\mu}W(2\text{-}\hat{\mu})$
Larsen Product, R_{LP}	$\hat{\mu}.\mu(output)$	$\hat{\mu}W$
Drastic Product, R_{DP}	Case 1: $\hat{\mu}$, if $\mu(output) = 1$ Case 2: $\mu(output)$ if $\hat{\mu} = 1$ Case 3: 0 if $\hat{\mu}, \mu(output)$>0	– W 0

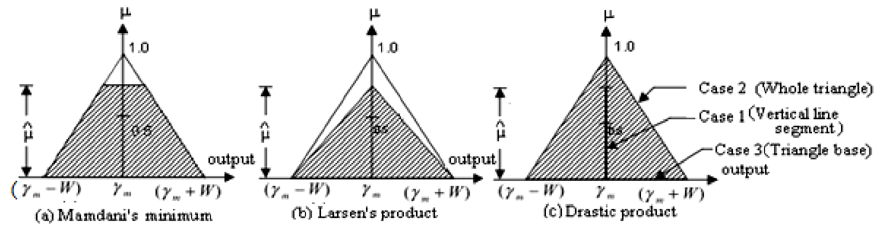

Figure 4. Illustration of inference methods

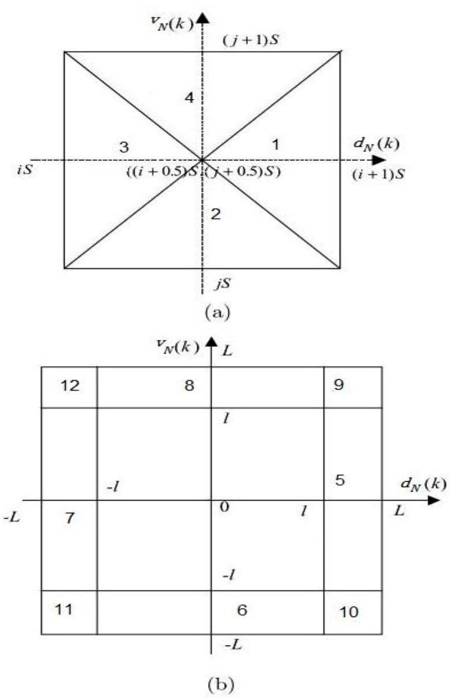

Figure 5. Possible input combinations of $d_N(kT)$ and $v_N(kT)$ (a) in the region: $iS \leq d_N\ (kT) \leq (i+1)S$ and $jS \leq v_N\ (kT) \leq (j+1)S$ (b) outside the region: $iS \leq d_N\ (kT) \leq (i+1)S$ and $jS \leq v_N\ (kT) \leq (j+1)S$

2.4. Inference engine

The degree of match between the crisp input and the fuzzy sets describing the meaning of the rule-antecedent is computed for each rule using minimum triangular norm defined in (17). Then the degree of match is used to determine the inferred output fuzzy set via any of the fuzzy inference methods defined in Table 1, and graphically illustrated in Figure 4.

There are four possible input combinations (ICs), see Figure 5(a), of the normalized inputs $d_N(kT)$ and $v_N(kT)$ in the region defined by $iS \leq d_N\ (kT) \leq (i+1)S$ and $jS \leq\ v_N\ (kT) \leq (j+1)S$. Similarly there are eight possible input combinations of the normalized inputs in the region shown in Figure 5(b).

The control rules in Section 2.3 are used to evaluate appropriate control law in each IC region. The outcomes of the control rules for all the IC regions with minimum triangular norm are listed in Table 2. It may be noticed from the control rule base that the control rules R_1 and R_3 fire the same output fuzzy set O_{m+1}, and produce two membership functions $\hat{\mu}_1(output)$ and $\hat{\mu}_3(output)$. To take care of the OR operation between rule 1 and rule 3, algebraic sum triangular co-norm is used, which is defined as follows:

$$\hat{\mu}_{1/3} = \hat{\mu}_1 + \hat{\mu}_3 - \hat{\mu}_1\ \hat{\mu}_3 \tag{18}$$

2.5. Defuzzification

The well-known COS method of defuzzification is used to obtain the crisp controller output which is defined [10] as

$$\Delta u_N(kT) = \frac{A(\hat{\mu}_1)h_1 + A(\hat{\mu}_2)h_2 + A(\hat{\mu}_3)h_3 + A(\hat{\mu}_4)h_4}{A(\hat{\mu}_1) + A(\hat{\mu}_2) + A(\hat{\mu}_3) + A(\hat{\mu}_4)} \tag{19}$$

where h_i, i=1, 2, 3 and 4 is the centroid of inferred output fuzzy set corresponding to the i^{th} rule. $\Delta u_N(kT)$ is replaced by $u_N(kT)$ for fuzzy PD controller. From the control rule base, it can be seen that the output fuzzy set, O_{m+1} is fired two times (see rules R_1 and R_3). In such a situation, using algebraic sum triangular co-norm defined in (18), (19) can be written as

$$\Delta u_N(kT) = \frac{A(\hat{\mu}_2)h_2 + A(\hat{\mu}_{1/3})h_{1/3} + A(\hat{\mu}_4)h_4}{A(\hat{\mu}_2) + A(\hat{\mu}_{1/3}) + A(\hat{\mu}_4)} \quad (20)$$

where $\hat{\mu}_{1/3}$ is the outcome of triangular co-norm and h is the centroid of the inferred output fuzzy set shown (with hatching) in Figure 4.

Since, the output fuzzy sets are symmetrical about their central values γ_m's, the global centroid can be calculated from the local centroids. The inferred area of respective output fuzzy set weighs the importance of each local centroid in the global centroid. Considering each of the local centroids shown in Figure 6, (20) can be written as

$$\Delta u(kT) = \left(\frac{1}{N_{\Delta u}}\right) \frac{A(\hat{\mu}_2)(mW) + A(\hat{\mu}_{1/3})(m+1)W + A(\hat{\mu}_4)(m+2)W}{A(\hat{\mu}_2) + A(\hat{\mu}_{1/3}) + A(\hat{\mu}_4)}$$

$$= \left(\frac{1}{N_{\Delta u}}\right) \frac{A(\hat{\mu}_2)(i+j)W + A(\hat{\mu}_{1/3})(i+j+1)W + A(\hat{\mu}_4)(i+j+2)W}{A(\hat{\mu}_2) + A(\hat{\mu}_{1/3}) + A(\hat{\mu}_4)} \quad (21)$$

Table 2. Outcomes of rules in IC regions

ICs	(R1) $\hat{\mu}_1$	(R2) $\hat{\mu}_2$	(R3) $\hat{\mu}_3$	(R4) $\hat{\mu}_4$
IC_1	μ_{V_j}	μ_{D_i}	μ_{D_i}	$\mu_{V_{j+1}}$
IC_2	$\mu_{D_{i+1}}$	μ_{D_i}	$\mu_{V_{j+1}}$	$\mu_{V_{j+1}}$
IC_3	$\mu_{D_{i+1}}$	μ_{V_j}	$\mu_{V_{j+1}}$	$\mu_{D_{i+1}}$
IC_4	μ_{V_j}	μ_{V_j}	μ_{D_i}	$\mu_{D_{i+1}}$
IC_5	$\mu_{V_{-J}}$	0	0	μ_{V_J}
IC_6	μ_{D_J}	$\mu_{D_{-J}}$	0	0
IC_7	0	$\mu_{V_{-J}}$	μ_{V_J}	0
IC_8	0	0	$\mu_{D_{-J}}$	μ_{D_J}
IC_9	1	0	0	0
IC_{10}	0	1	0	0
IC_{11}	0	0	1	0
IC_{12}	0	0	0	1

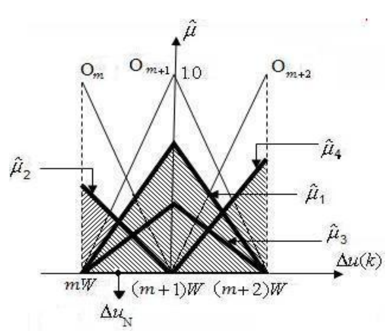

Figure 6. Center of sums defuzzification method (for R_{LP} inference method)

3. Analytical structures of the fuzzy PI controllers with symmetric triangular fuzzy sets

Here analytical structures of three classes of fuzzy PI controllers with multiple symmetric fuzzy sets are presented. The expression in (21) can be written as

$$\Delta u(kT) = \frac{(i+j+1)W}{N_{\Delta u}} + \left(\frac{W}{N_{\Delta u}}\right) \frac{A(\hat{\mu}_4) - A(\hat{\mu}_2)}{A(\hat{\mu}_2) + A(\hat{\mu}_{1/3}) + A(\hat{\mu}_4)}$$

$$= \Delta u_g + \Delta u_l \quad (22)$$

where $$\Delta u_g = \frac{(i+j+1)W}{N_{\Delta u}} \quad (23)$$

and $$\Delta u_l = \left(\frac{W}{N_{\Delta u}}\right) \frac{A(\hat{\mu}_4) - A(\hat{\mu}_2)}{A(\hat{\mu}_2) + A(\hat{\mu}_{1/3}) + A(\hat{\mu}_4)} \quad (24)$$

For simplicity we define

$$z_1(k) = d_N(k) - (i + 0.5)S \quad (25)$$

$$z_2(k) = v_N(k) - (j + 0.5)S \quad (26)$$

and show kT as k in the sequel. The analytical structures of different classes of fuzzy PI controllers follow now.

3.1. Analytical structures in the regions IC_1 - IC_4

Class I: Triangular norm: minimum, triangular co-norm: algebraic sum, inference method: Mamdani minimum.

$$\Delta u_l(k) = \left(\frac{W}{N_{\Delta u}}\right) \frac{(S + |z_1(k) - z_2(k)|)(z_1(k) + z_2(k))}{3S^2 - 2S|z(k)| - 2\left(z_1^2(k) + z_2^2(k)\right) - \frac{W}{S^2}(0.5625\,S^4 - 0.75S^3|z_1(k) + z_2(k)| - 0.75S^2(z_1^2(k) + z_2^2(k)) + S^2 z_1(k) z_2(k) + S z_1(k) z_2(k)|z_1(k) + z_2(k)| + z_1^2(k) z_2^2(k))} \quad (27)$$

where $z(k)$ is defined in Table 3.

Class II: Triangular norm: minimum, triangular co-norm: algebraic sum, inference method: Larsen product.

$$\Delta u_l(k) = \left(\frac{W}{N_{\Delta u}}\right) \frac{z_1(k) + z_2(k)}{2S - 2|z(k)| - \frac{W}{S}(0.25S^2 - 0.5S|z_1(k) + z_2(k)| + z_1(k) z_2(k))} \quad (28)$$

where $z(k)$ is defined in Table 3.

Class III: Triangular norm: minimum, triangular co-norm: algebraic sum, inference method: drastic product.

$$\Delta u_l(k) = \left(\frac{W}{N_{\Delta u}}\right) \frac{z_1(k) + z_2(k)}{2S - 2|z(k)| - \frac{1}{S}(0.25S^2 - 0.5S|z_1(k) + z_2(k)| + z_1(k) z_2(k))} \quad (29)$$

where $z(k)$ is defined in Table 3.

Table 3. Attributes of $z(k)$

$z(k)$	ICs
$z_1(k)$	IC_1 , IC_3
$z_2(k)$	IC_2 , IC_4

For all classes, Δu_g in regions IC_1- IC_4 is given by (23).

3.2. Analytical structures in the regions IC_5 – IC_8

Class I, class II and class III:

$$\Delta u_g = \frac{(J+j)W}{N_{\Delta u}} \quad \text{in the region } IC_5 \qquad (30)$$

$$\Delta u_g = \frac{(i-J+1)W}{N_{\Delta u}} \quad \text{in the region } IC_6 \qquad (31)$$

$$\Delta u_g = \frac{(-J+j+1)W}{N_{\Delta u}} \quad \text{in the region } IC_7 \qquad (32)$$

$$\Delta u_g = \frac{(i+J)W}{N_{\Delta u}} \quad \text{in the region } IC_8 \qquad (33)$$

Class I:

$$\Delta u_l(k) = \frac{W}{2N_{\Delta u}}\left[\frac{Sz_2(k)}{0.75S^2-z_2^2(k)}+1\right] \quad \text{in the region } IC_5 \qquad (34)$$

$$\Delta u_l(k) = \frac{W}{2N_{\Delta u}}\left[\frac{Sz_1(k)}{0.75S^2-z_1^2(k)}-1\right] \quad \text{in the region } IC_6 \qquad (35)$$

$$\Delta u_l(k) = \frac{W}{2N_{\Delta u}}\left[\frac{Sz_2(k)}{0.75S^2-z_2^2(k)}-1\right] \quad \text{in the region } IC_7 \qquad (36)$$

$$\Delta u_l(k) = \frac{W}{2N_{\Delta u}}\left[\frac{Sz_1(k)}{0.75S^2-z_1^2(k)}+1\right] \quad \text{in the region } IC_8 \qquad (37)$$

Class II and class III:

$$\Delta u_l(k) = \frac{W}{2N_{\Delta u}}\left[\frac{2z_2(k)}{S}+1\right] \quad \text{in the region } IC_5 \qquad (38)$$

$$\Delta u_l(k) = \frac{W}{2N_{\Delta u}}\left[\frac{2z_1(k)}{S}-1\right] \quad \text{in the region } IC_6 \qquad (39)$$

$$\Delta u_l(k) = \frac{W}{2N_{\Delta u}}\left[\frac{2z_2(k)}{S}-1\right] \quad \text{in the region } IC_7 \qquad (40)$$

$$\Delta u_l(k) = \frac{W}{2N_{\Delta u}}\left[\frac{2z_1(k)}{S}+1\right] \quad \text{in the region } IC_8 \qquad (41)$$

3.3. Analytical structures in the regions IC_9 - IC_{12}

Class I, class II and class III:

$$\Delta u(k) = 0 \quad \text{in the region } IC_9 \text{ and } IC_{11} \qquad (42)$$

$$\Delta u(k) = \frac{-H}{N_{\Delta u}} \quad \text{in the region } IC_{10} \qquad (43)$$

$$\Delta u(k) = \frac{H}{N_{\Delta u}} \quad \text{in the region } IC_{12} \qquad (44)$$

4. Properties of fuzzy PI controllers with symmetric triangular fuzzy sets

In (22) Δu_g represents the global two-dimensional multi-level relay and $\Delta u_l(k)$ represents the local nonlinear PI controller. In order to examine the roles of global multilevel relay and local nonlinear PI controller in total control action, and the degree of nonlinearity of fuzzy controller as N changes, we define a constant η as

$$\eta = \frac{|\Delta u_l|_{max}}{|\Delta u_g|_{max} + |\Delta u_l|_{max}} \qquad (45)$$

It can be seen from (45) that (i) as $N \geq 3$, for $N = 3$ the value of η is 50% which implies that the global multilevel relay and the local nonlinear PI controller play equal role in total control action, and (ii) as N becomes larger, η approaches a smaller value which makes the resolution of global multi-level relay output finer and the fuzzy controller less nonlinear. Moreover, the global multi-level relay has a stronger role than the local nonlinear PI controller, in the total control action.
In view of the above analytical structures and their properties we have

Theorem 1: *The structure of the fuzzy PI controller with linear control rules is the sum of a global two-dimensional multi-level relay and a local nonlinear PI controller.*

5. Analytical structures of fuzzy PD controllers

Since it is clear from Figure 1 that $d(kT)$ and $v(kT)$ are always the inputs to the two-term (PI or PD) controller and $\Delta u(kT)$ is the incremental output of the PI controller (velocity algorithm) while $u(kT)$ is the output of the PD controller, the analytical structures for fuzzy PI controllers in Section 3 are equally applicable to even fuzzy PD controllers provided we replace $\Delta u(kT)$ by $u(kT)$, $N_{\Delta u}^{-1}$ by N_u^{-1} and set $u[(k-1)T]$ equal to zero. Also, the properties of different classes of fuzzy PI controllers in Section 4 hold good for fuzzy PD controllers.

6. BIBO stability analysis of feedback systems that contain fuzzy PD controllers

We make use of the well-known small-gain theorem in [1] to derive sufficient conditions for BIBO stability of feedback systems which contain one of the fuzzy PD controllers in Section 3 as a subsystem. The sufficient condition for fuzzy PD control system to be BIBO stable can be stated as follows.

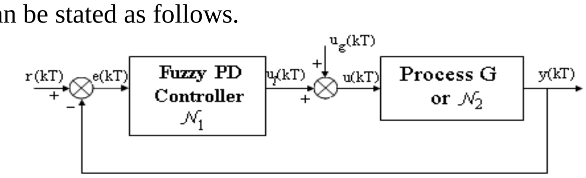

Figure 7. Block diagram of a typical fuzzy PD control system

Theorem 2: *The system of Figure 7 is BIBO stable if (i) the given nonlinear process N_2 has a bounded norm i.e., $|N_2| < \infty$ where $\|.\|$ denotes the norm of nonlinear mapping, and (ii) the parameters S, W, T, N_d, N_v and N_u of fuzzy PD controller satisfy the inequality*

$$\frac{W \times (T \times N_d + N_v)}{S \times T \times N_u} \|N_2\| < 1 \qquad (46)$$

This theorem is applicable to all the three classes of control systems.

7. Validation of fuzzy PD controller

Comparison of the performances of linear PD controller and the fuzzy PD controller with symmetric input and output fuzzy sets is done here by considering the following example

A linear second-order time-delay system

$$G_p(s) = \frac{e^{-3s}}{(100s+1)^2} \quad (47)$$

with a unit setpoint. In (47), $G_p(s)$ represent the transfer function of the plant to be controlled. To implement the mathematical model of fuzzy PD controller, the design parameters are to be appropriately chosen. For this, the design procedure in [12] is followed here to design the fuzzy PD controller. For the above process, the values of sampling period T=0.1sec, proportional gain $K_P^d = 6.0$, derivative gain $K_D^d = 2.5$, maximum absolute displacement (error) $|d|_{max} = 1$ and maximum absolute velocity $|v|_{max} = 0.0151$. Since the control performance with the calculated design parameters is not found to be satisfactory, the parameters of the fuzzy controller are then fine tuned on trial and error basis to attain acceptable performance. The parameters N, N_d, N_v, N_u, l and H of the fuzzy controllers are listed in Table 4.

The unit step responses obtained with different classes of fuzzy PD controllers for the plant $G_p(s)$ are shown in Figure 8. These figures also show the corresponding responses with the conventional PD controllers. From the plots it is apparent that the fuzzy PD controllers outperform the conventional PD controllers for different values of N. The basic objective of this simulation study is to demonstrate the influence of the parameter N on the performance of the fuzzy controller. Therefore, for different classes of fuzzy PD controllers other functional parameters are fixed and only N is varied as shown in Table 4.

From the time-domain performance data in Table 4 it is observed that for class I and class III fuzzy PD controllers, by increasing the value of N improved performance is obtained with drastic fall in the value of percent peak overshoot ($\%M_p$) and slight variation in the values of rise time (t_r) and settling time (t_s).

8. Conclusions

In this paper, analytical structures for different classes of fuzzy PI/PD controllers with multiple fuzzy sets have been considered. Using N_1 number of symmetric triangular membership functions on the input variable 'displacement', N_2 number of symmetric triangular membership functions on the input variable 'velocity', N_1+$N_2 - 1$ number of symmetric triangular fuzzy sets for the output variable, linear control rules, minimum triangular norm, algebraic sum triangular co-norm, three different inference methods (R_{MM}, R_{LP}, and R_{DP}) and COS defuzzification, expressions for control laws of fuzzy PI/PD controllers have been derived. It has been shown that each resulting controller is equivalent to the sum of a global two- dimensional multilevel relay and a local nonlinear PI/PD controller.

Upon carefully investigating the properties of the fuzzy controllers, it has been found that all the three classes of fuzzy controllers exhibit desirable control properties. Using the small-gain theorem, BIBO stability conditions are established considering any class of the controller as a fuzzy PD controller. The superiority of fuzzy PD controller over the linear PD controller has been demonstrated through a simulation study on a linear second-order time-delay system.

Table 4. Attributes and time-domain performance data of plants with conventional and fuzzy PD controllers

Plant	Controller	Class	N_d	N_v	N_u	l	H	N	M_p (%)	t_r (sec)	t_s (sec)
$G_p(s)$	Linear PD	-	-	-	-	-	-	-	27.94	59.1365	-
	Fuzzy PD	I	2.4	34.0	0.022	2.4	4.0	3	3.1833	14.3	20.4
		I	2.4	34.0	0.022	2.4	4.0	5	2.4623	16.2	22.4
		I	2.4	34.0	0.022	2.4	4.0	7	2.2740	15.2	21.4
		II	2.56	26.0	0.0085	3.2	4.0	3	6.9945	10.0	15.0
		II	2.56	26.0	0.0085	3.2	4.0	5	2.3737	10.0	15.1
		II	2.56	26.0	0.0085	3.2	4.0	7	2.4792	10.0	14.9
		III	2.21	24.0	0.006	3.0	3.0	3	4.655	11.0	16.0
		III	2.21	24.0	0.006	3.0	3.0	5	0.0020	11.0	17.0
		III	2.21	24.0	0.006	3.0	3.0	7	0.0001	11.0	17.0

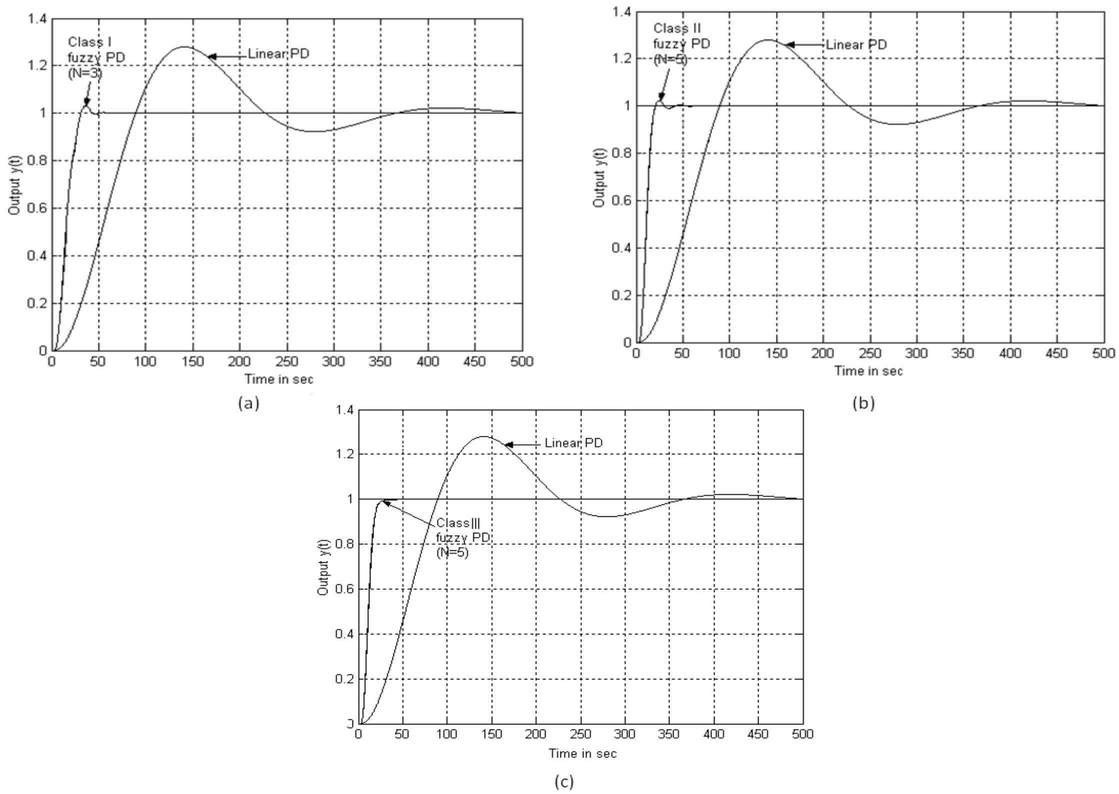

Figure 8. Unit step response of the closed loop system with plant $G_p(s)$ and (a) Class I (b) Class II(c) Class III fuzzy PD controller

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